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Last modified Tuesday December 11, 2001 by: [Sape A. Miedema](#)

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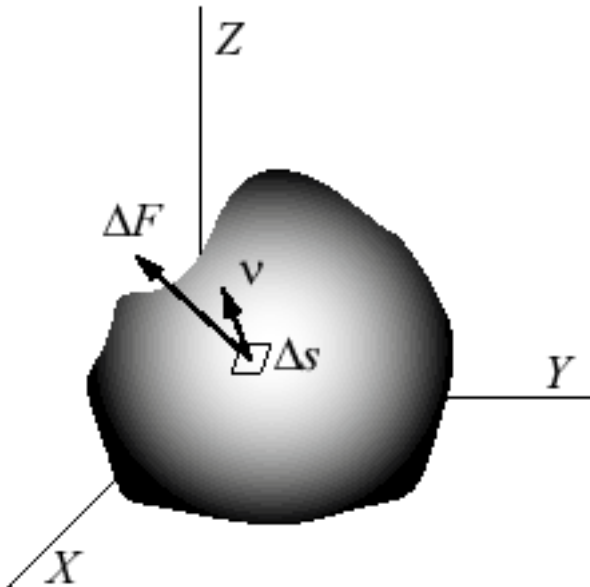


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Mechanics of Materials: Stress

The Definition of Stress

The concept of **stress** originated from the study of strength and failure of solids. The stress field is the distribution of internal "tractions" that balance a given set of external tractions and body forces.



First, we look at the external traction \mathbf{T} that represents the force per unit area acting at a given location on the body's surface. Traction \mathbf{T} is a *bound vector*, which means \mathbf{T} cannot slide along its line of action or translate to another location and keep the same meaning.

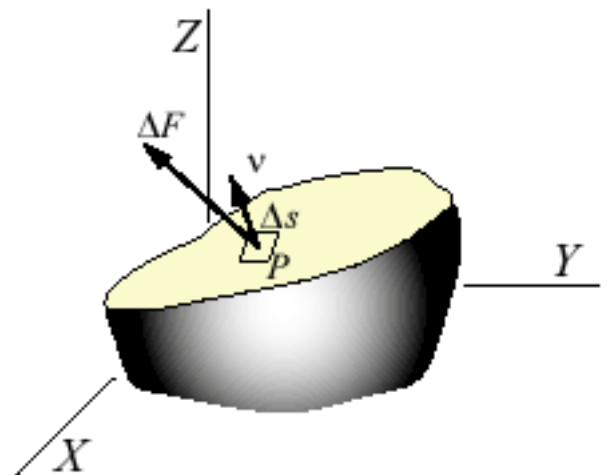
In other words, a traction vector cannot be fully described unless both the force and the surface where the force acts on has been specified. Given both ΔF and Δs , the traction \mathbf{T} can be defined as

$$\mathbf{T} = \lim_{\Delta s \rightarrow 0} \frac{\Delta \mathbf{F}}{\Delta s} = \frac{d\mathbf{F}}{ds}$$

The internal traction within a solid, or stress, can be defined in a similar manner. Suppose an arbitrary slice is made across the solid shown in the above figure, leading to the free body diagram shown at right.

Surface tractions would appear on the exposed surface, similar in form to the external tractions applied to the body's exterior surface. The stress at point P can be defined using the same equation as was used for \mathbf{T} .

Stress therefore can be interpreted as internal tractions that act on a defined internal datum plane. One cannot measure the stress without first specifying the datum plane.



The Stress Tensor (or Stress Matrix)

Surface tractions, or stresses acting on an internal datum plane, are typically decomposed into three mutually orthogonal components. One component is normal to the surface and represents *direct stress*. The other two components are tangential to the surface and represent *shear stresses*.

What is the distinction between normal and tangential tractions, or equivalently, direct and shear stresses? **Direct stresses** tend to change the volume of the material (e.g. hydrostatic pressure) and are resisted by the body's bulk modulus (which depends on the Young's modulus and Poisson ratio). **Shear stresses** tend to deform the material without changing its volume, and are resisted by the body's shear modulus.

Defining a set of internal datum planes aligned with a Cartesian coordinate system allows the stress state at an internal point P to be described relative to x , y , and z coordinate directions.

For example, the stress state at point P can be represented by an *infinitesimal* cube with three stress components on each of its six sides (one direct and two shear components).

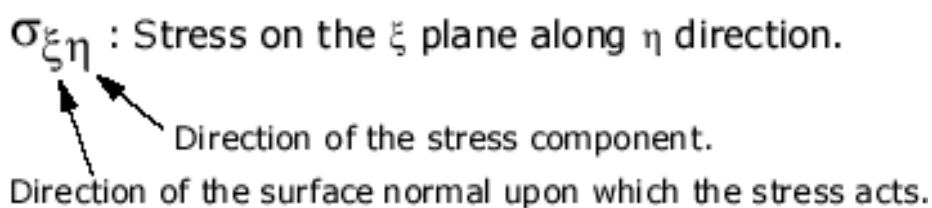
Since each point in the body is under static equilibrium (no net force in the absence of any body forces), only nine stress components from three planes are needed to describe the stress state at a point P .

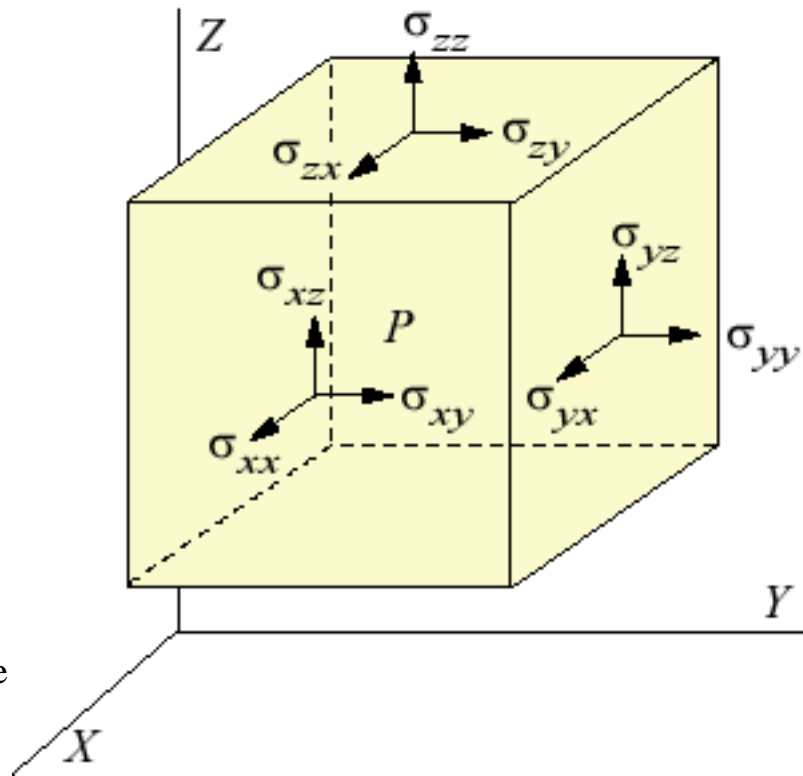
These nine components can be organized into the matrix:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

where shear stresses across the diagonal are identical (i.e. $\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{zy}$, and $\sigma_{zx} = \sigma_{xz}$) as a result of static equilibrium (no net moment). This grouping of the nine stress components is known as the **stress tensor** (or stress matrix).

The subscript notation used for the nine stress components have the following meaning:

$\sigma_{\xi\eta}$: Stress on the ξ plane along η direction.




Note: The stress state is a second order tensor since it is a quantity associated with two directions. As a result, stress components have 2 subscripts.

A surface traction is a first order tensor (i.e. vector) since it a quantity associated with only one direction. Vector components therefore require only 1 subscript.

Mass would be an example of a zero-order tensor (i.e. scalars), which have no relationships with directions (and no subscripts).

Equations of Equilibrium

Consider the static equilibrium of a solid subjected to the body force vector field \mathbf{b} . Applying Newton's first law of motion results in the following set of differential equations which govern the stress distribution within the solid,

$$\begin{cases} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + b_x = 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + b_y = 0 \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0 \end{cases}$$

In the case of two dimensional stress, the above equations reduce to,

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + b_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0 \end{cases}$$

Plane Stress and Coordinate Transformations

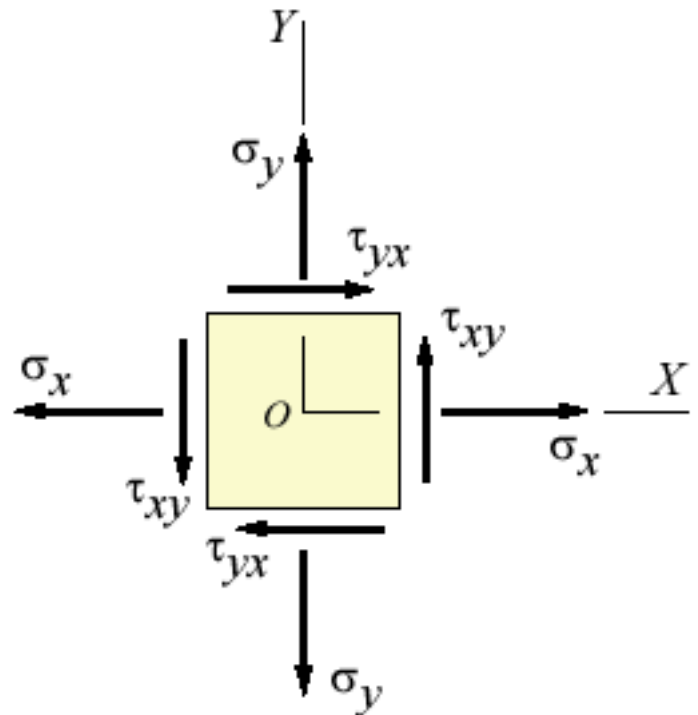
Plane State of Stress

A class of common engineering problems involving stresses in a thin plate or on the free surface of a structural element, such as the surfaces of thin-walled pressure vessels under external or internal pressure, the free surfaces of shafts in torsion and beams under transverse load, have one [principal stress](#) that is much smaller than the other two. By assuming that this small principal stress is zero, the three-dimensional stress state can be reduced to two dimensions. Since the remaining two principal stresses lie in a plane, these simplified 2D problems are called **plane stress** problems.

Assume that the negligible principal stress is oriented in the z -direction. To reduce the [3D stress matrix](#) to the 2D plane stress matrix, remove all components with z subscripts to get,

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

where $\tau_{xy} = \tau_{yx}$ for static equilibrium. The sign convention for positive stress components in plane stress is illustrated in the above figure on the 2D element.



Coordinate Transformations

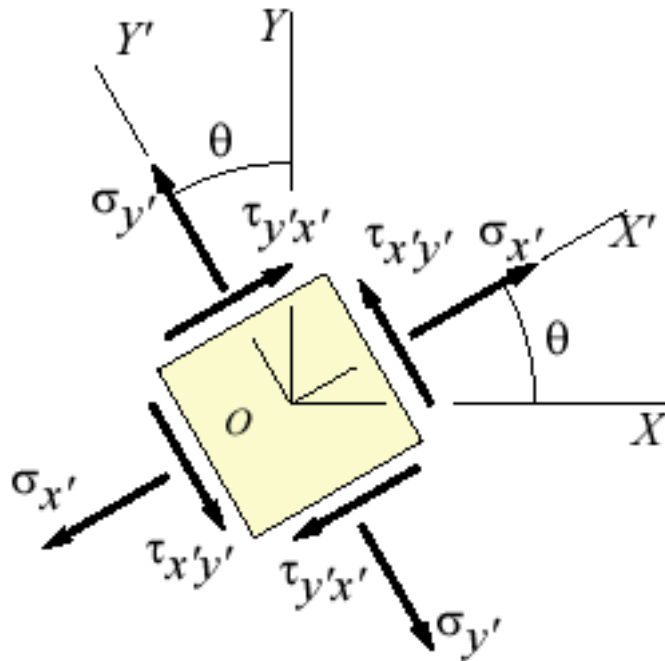
The coordinate directions chosen to analyze a structure are usually based on the shape of the structure. As a result, the direct and shear stress components are associated with these directions. For example, to analyze a bar one almost always directs one of the coordinate directions along the bar's axis.

Nonetheless, stresses in directions that do not line up with the original coordinate set are also important. For example, the failure plane of a brittle shaft under torsion is often at a 45° angle with respect to the shaft's axis. Stress transformation formulas are required to analyze these stresses.

The transformation of stresses with respect to the $\{x,y,z\}$ coordinates to the stresses with respect to $\{x',y',z'\}$ is performed via the equations,

$$\left\{ \begin{array}{l} \sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\ \sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \\ \quad = \sigma_x + \sigma_y - \sigma_{x'} \\ \tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \end{array} \right.$$

where θ is the rotation angle between the two coordinate sets (positive in the counterclockwise direction). This angle along with the stresses for the $\{x',y',z'\}$ coordinates are shown in the figure below,



Principal Stress for the Case of Plane Stress

Principal Directions, Principal Stress

The normal stresses ($\sigma_{x'}$ and $\sigma_{y'}$) and the shear stress ($\tau_{x'y'}$) vary smoothly with respect to the rotation angle θ , in accordance with the [coordinate transformation](#) equations. There exist a couple of particular angles where the stresses take on special values.

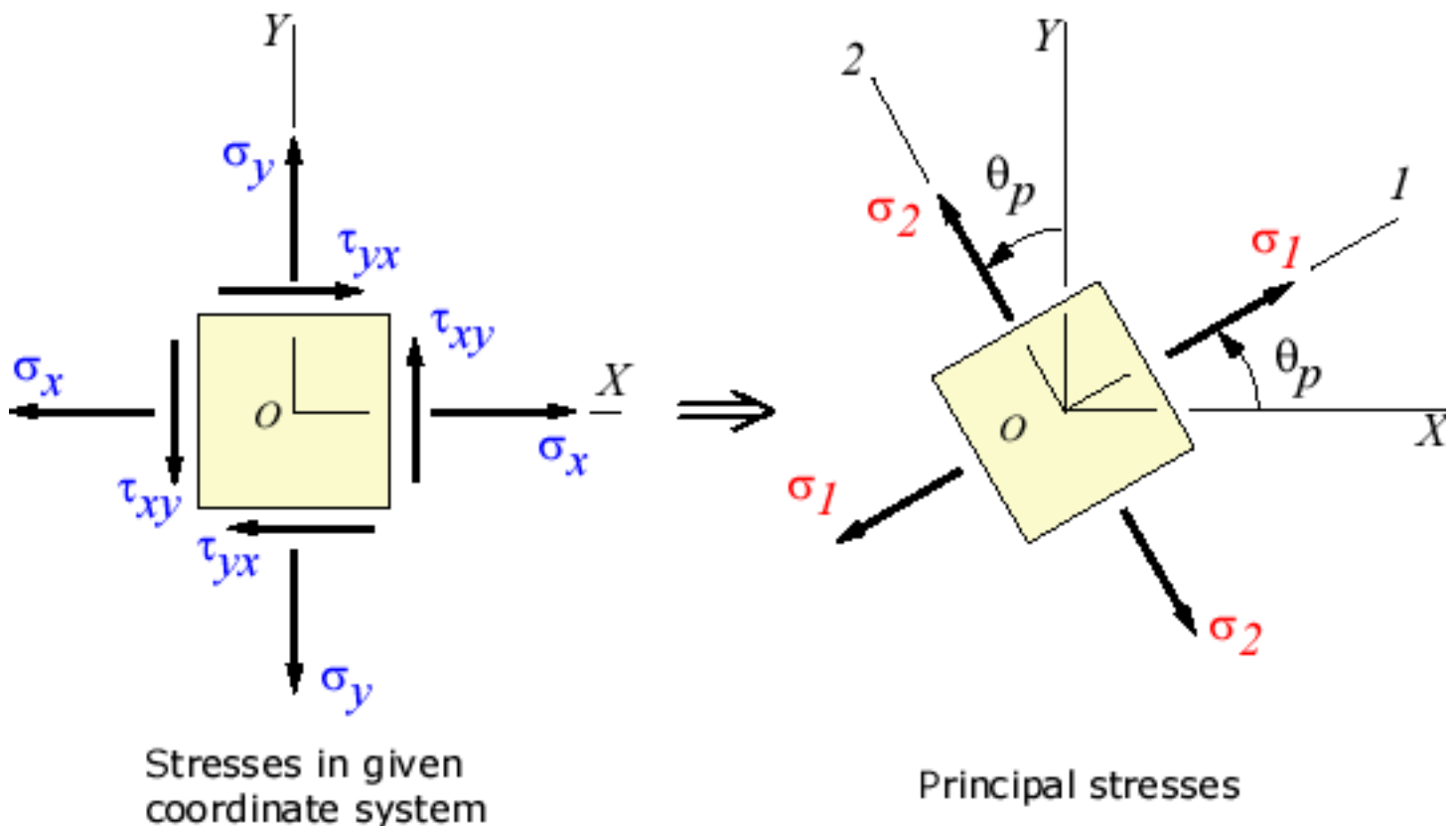
First, there exists an angle θ_p where the shear stress $\tau_{x'y'}$ becomes zero. That angle is found by setting $\tau_{x'y'}$ to zero in the above shear transformation equation and solving for θ (set equal to θ_p). The result is,

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

The angle θ_p defines the *principal directions* where the only stresses are normal stresses. These stresses are called *principal stresses* and are found from the original stresses (expressed in the x,y,z directions) via,

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

The transformation to the principal directions can be illustrated as:



Maximum Shear Stress Direction

Another important angle, θ_s , is where the maximum shear stress occurs. This is found by finding the maximum of the shear stress transformation equation, and solving for θ . The result is,

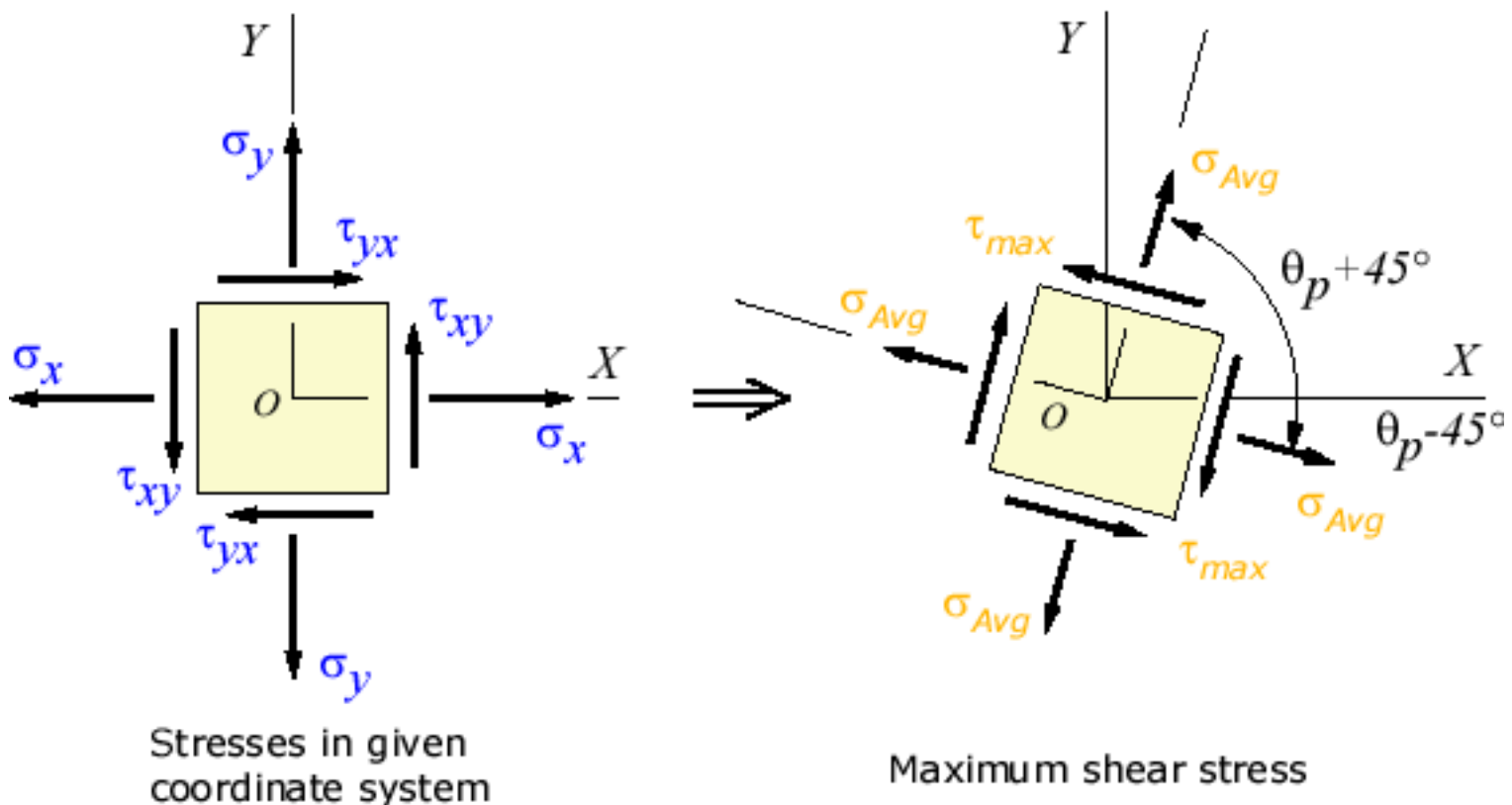
$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\Rightarrow \theta_s = \theta_p \pm 45^\circ$$

The maximum shear stress is equal to one-half the difference between the two principal stresses,

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_1 - \sigma_2}{2}$$

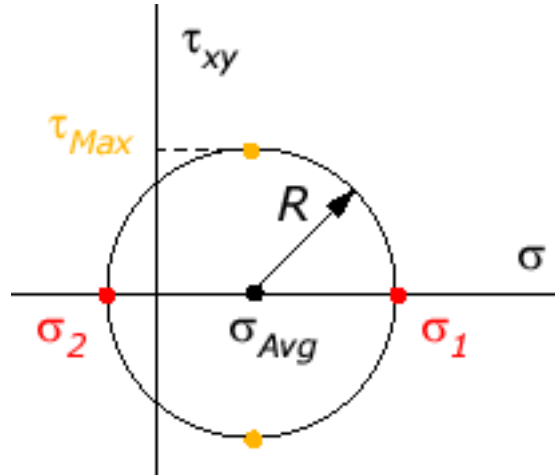
The transformation to the maximum shear stress direction can be illustrated as:



Mohr's Circle for Plane Stress

Mohr's Circle

Introduced by Otto Mohr in 1882, Mohr's Circle illustrates principal stresses and stress transformations via a graphical format,



The two principal stresses are shown in **red**, and the maximum shear stress is shown in **orange**. Recall that the normal stresses equal the principal stresses when the stress element is aligned with the principal directions, and the shear stress equals the maximum shear stress when the stress element is rotated 45° away from the principal directions.

As the stress element is rotated away from the [principal](#) (or maximum shear) directions, the normal and shear stress components will always lie on Mohr's Circle.

Mohr's Circle was the leading tool used to visualize relationships between normal and shear stresses, and to estimate the maximum stresses, before hand-held calculators became popular. Even today, Mohr's Circle is still widely used by engineers all over the world.

Derivation of Mohr's Circle

To establish Mohr's Circle, we first recall the stress [transformation formulas](#) for plane stress at a given location,

$$\begin{cases} \sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\ \tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \end{cases}$$

Using a [basic trigonometric relation](#) ($\cos^2 2\theta + \sin^2 2\theta = 1$) to combine the two above equations we have,

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

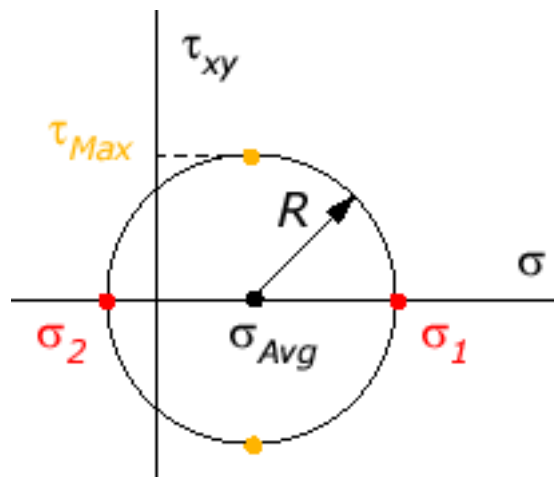
This is the equation of a circle, plotted on a graph where the abscissa is the normal stress and the ordinate is the shear stress. This is easier to see if we interpret σ_x and σ_y as being the two [principal stresses](#), and τ_{xy} as being the maximum shear stress. Then we can define the average stress, σ_{avg} , and a "radius" R (which is just equal to the maximum shear stress),

$$\sigma_{Avg} = \frac{\sigma_x + \sigma_y}{2} \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

The circle equation above now takes on a more familiar form,

$$\left(\sigma_{x'} - \sigma_{Avg}\right)^2 + \tau_{x'y'}^2 = R^2$$

The circle is centered at the average stress value, and has a radius R equal to the maximum shear stress, as shown in the figure below,



Related Topics

The procedure of drawing a Mohr's Circle from a given stress state is discussed in the [Mohr's Circle usage](#) page.

The Mohr's Circle for [plane strain](#) can also be obtained from similar procedures.

Mohr's Circle Usage in Plane Stress

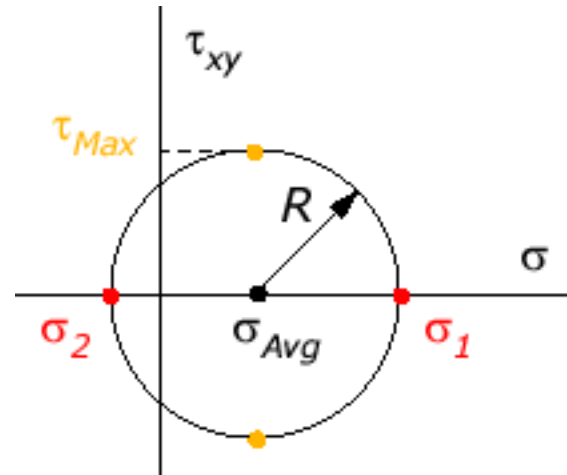
Principal Stresses from Mohr's Circle

A chief benefit of Mohr's circle is that the [principal stresses](#) σ_1 and σ_2 and the maximum shear stress τ_{max} are obtained immediately after drawing the circle,

$$\begin{cases} \sigma_{1,2} = \sigma_{Avg} \pm R \\ \tau_{Max} = R \end{cases}$$

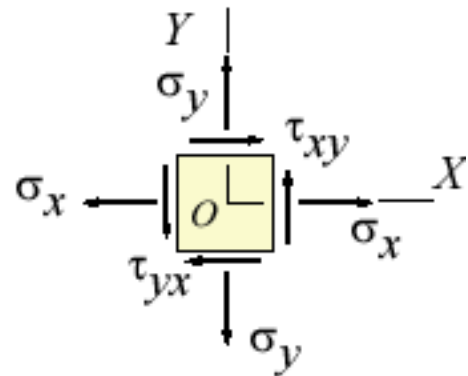
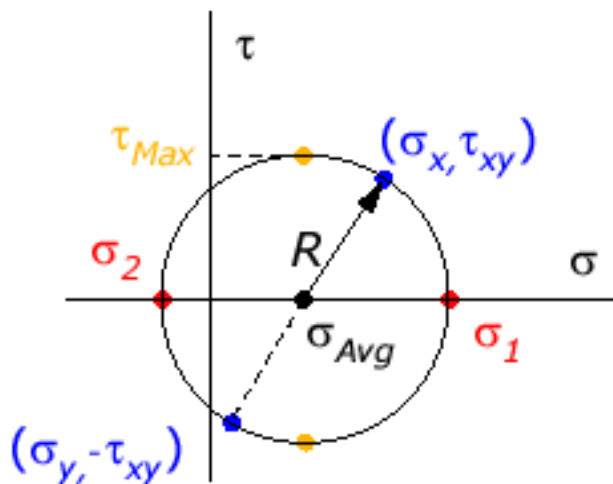
where,

$$\sigma_{Avg} = \frac{\sigma_x + \sigma_y}{2} \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



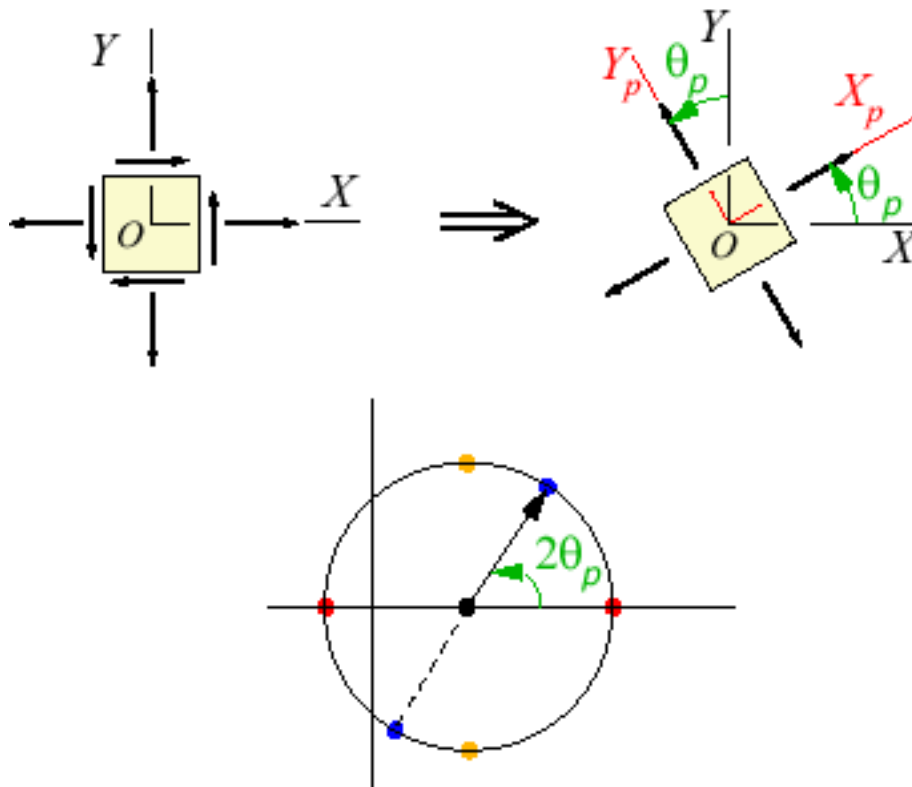
Principal Directions from Mohr's Circle

Mohr's Circle can be used to find the directions of the principal axes. To show this, first suppose that the normal and shear stresses, σ_x , σ_y , and τ_{xy} , are obtained at a given point O in the body. They are expressed relative to the coordinates XY , as shown in the stress element at right below.



The Mohr's Circle for this general stress state is shown at left above. Note that it's centered at σ_{avg} and has a radius R , and that the two points $\{\sigma_x, \tau_{xy}\}$ and $\{\sigma_y, -\tau_{xy}\}$ lie on opposite sides of the circle. The line connecting σ_x and σ_y will be defined as L_{xy} .

The **angle** between the current axes (X and Y) and the **principal axes** is defined as θ_p , and is equal to one half the angle between the line L_{xy} and the σ -axis as shown in the schematic below,



A set of six Mohr's Circles representing most stress state possibilities are presented on the [examples](#) page.

Also, principal directions can be computed by the [principal stress calculator](#).

Rotation Angle on Mohr's Circle

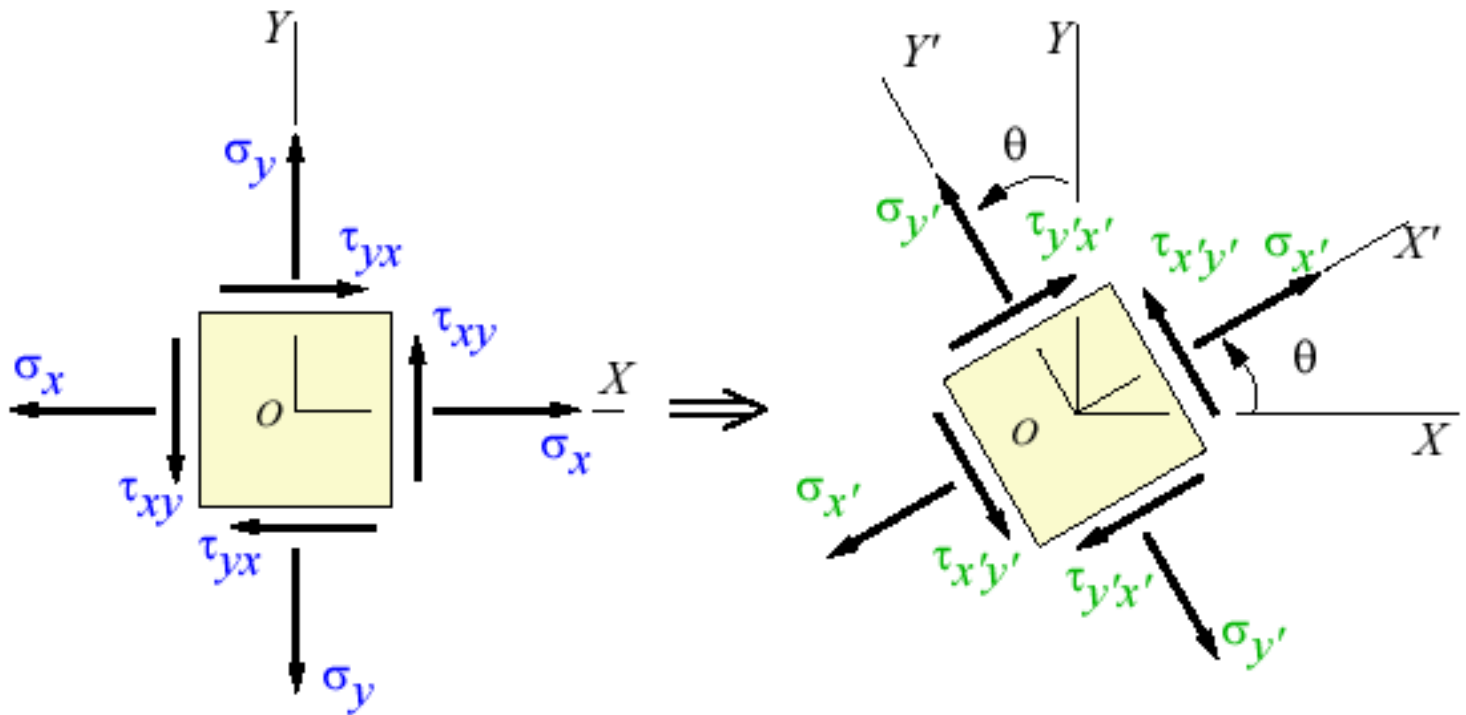
Note that the coordinate rotation angle θ_p is defined positive when starting at the XY coordinates and proceeding to the X_pY_p coordinates. In contrast, on the Mohr's Circle θ_p is defined positive starting on the principal stress line (i.e. the σ -axis) and proceeding to the XY stress line (i.e. line L_{xy}). The angle θ_p has the opposite sense between the two figures, because on one it starts on the XY coordinates, and on the other it starts on the principal coordinates.

Some books avoid this dichotomy between θ_p on Mohr's Circle and θ_p on the stress element by locating $(\sigma_x, -\tau_{xy})$ instead of (σ_x, τ_{xy}) on Mohr's Circle. This will switch the polarity of θ_p on the circle. Whatever method you choose, the bottom line is that an *opposite* sign is needed either in the interpretation or in the plotting to make Mohr's Circle physically meaningful.

Stress Transform by Mohr's Circle

Mohr's Circle can be used to transform stresses from one coordinate set to another, similar that that described on the [plane stress](#) page.

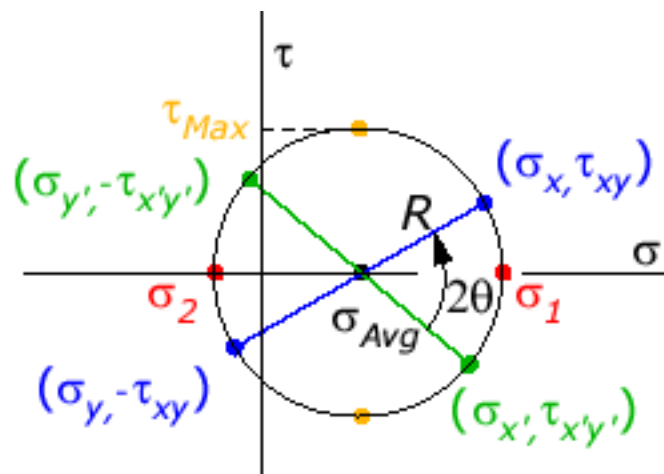
Suppose that the normal and shear stresses, σ_x , σ_y , and τ_{xy} , are obtained at a point O in the body, expressed with respect to the coordinates XY . We wish to find the stresses expressed in the new coordinate set $X'Y'$, rotated an angle θ from XY , as shown below:



Stresses at given coordinate system Stresses transformed to another coordinate

To do this we proceed as follows:

- Draw Mohr's circle for the **given stress state** (σ_x , σ_y , and τ_{xy} ; shown below).
- Draw the line L_{xy} across the circle from (σ_x, τ_{xy}) to $(\sigma_y, -\tau_{xy})$.
- Rotate the line L_{xy} by $2*\theta$ (twice as much as the angle between XY and $X'Y'$) and in the *opposite* direction of θ .
- The **stresses in the new coordinates** ($\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$) are then read off the circle.

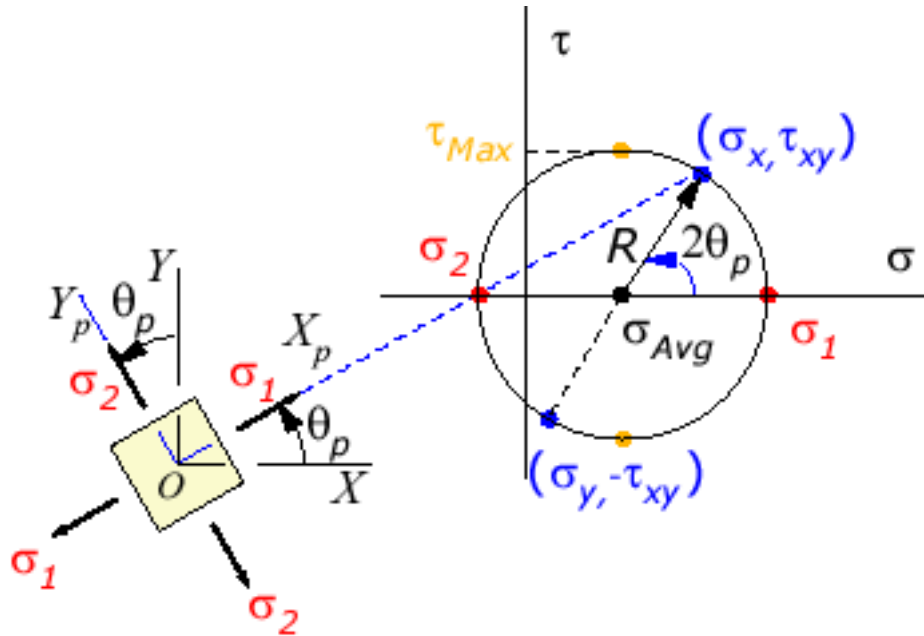


Stress transforms can be performed using eFunda's [stress transform calculator](#).

Examples of Mohr's Circles in Plane Stress

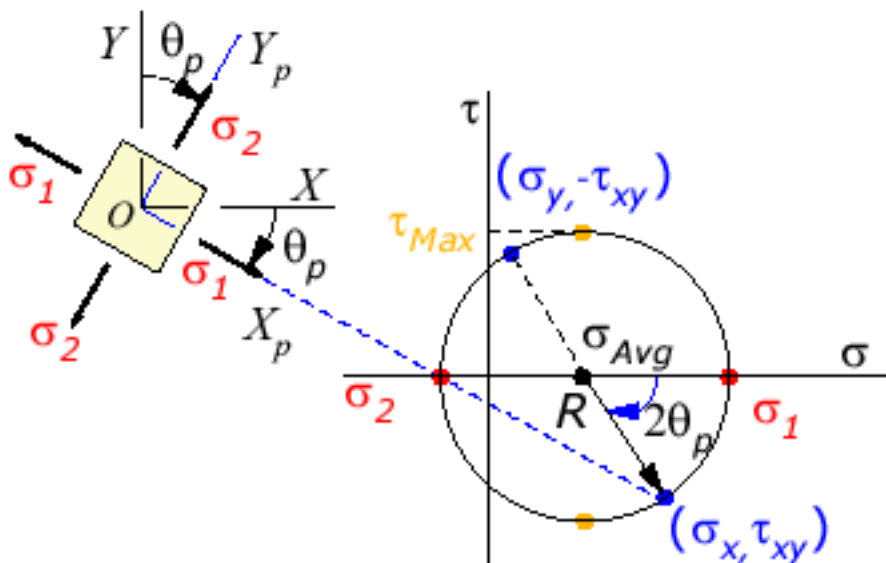
Case 1: $\tau_{xy} > 0$ and $\sigma_x > \sigma_y$

The principal axes are **counterclockwise** to the current axes (because $\tau_{xy} > 0$) and no more than 45° away (because $\sigma_x > \sigma_y$).



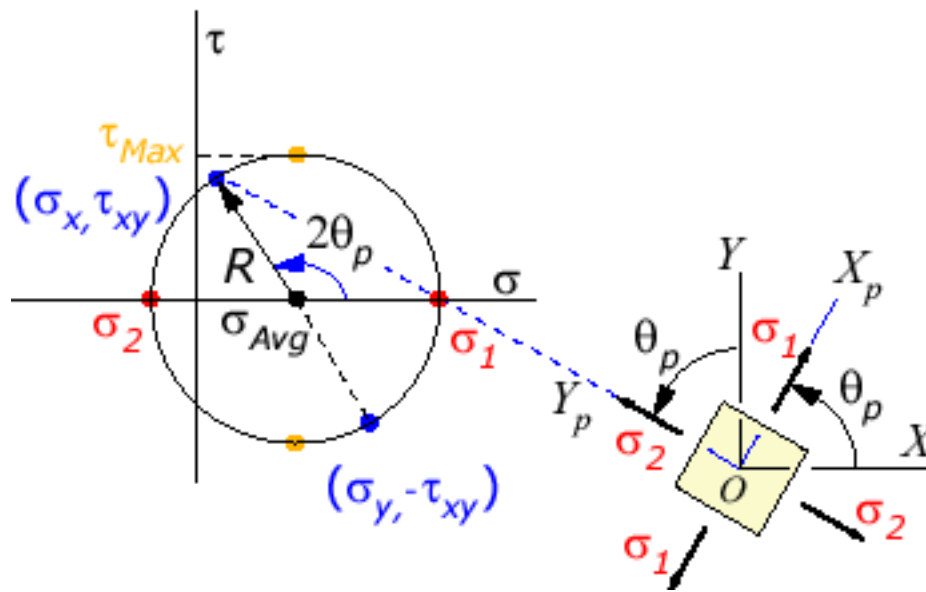
Case 2: $\tau_{xy} < 0$ and $\sigma_x > \sigma_y$

The principal axes are **clockwise** to the current axes (because $\tau_{xy} < 0$) and no more than 45° away (because $\sigma_x > \sigma_y$).



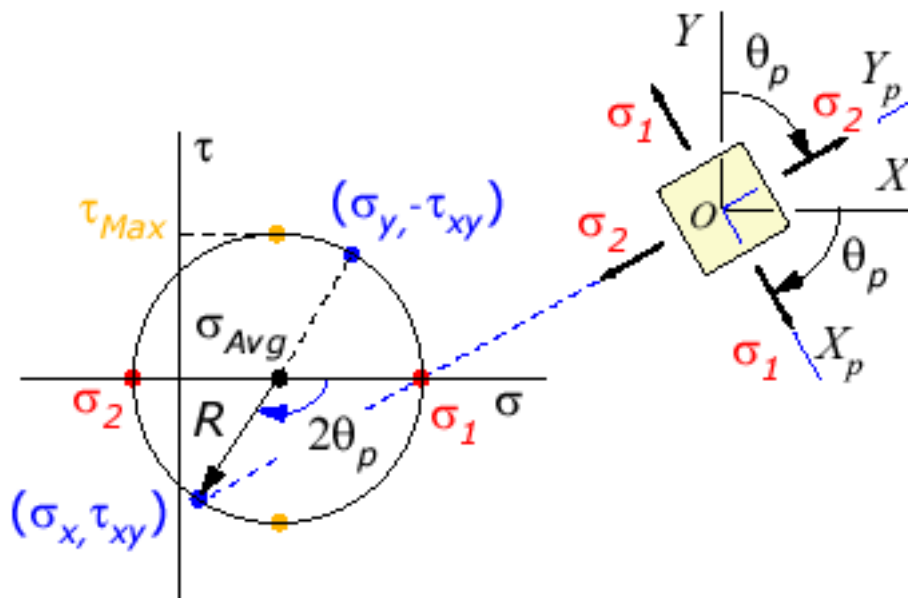
Case 3: $\tau_{xy} > 0$ and $\sigma_x < \sigma_y$

The principal axes are **counterclockwise** to the current axes (because $\tau_{xy} > 0$) and between 45° and 90° away (because $\sigma_x < \sigma_y$).



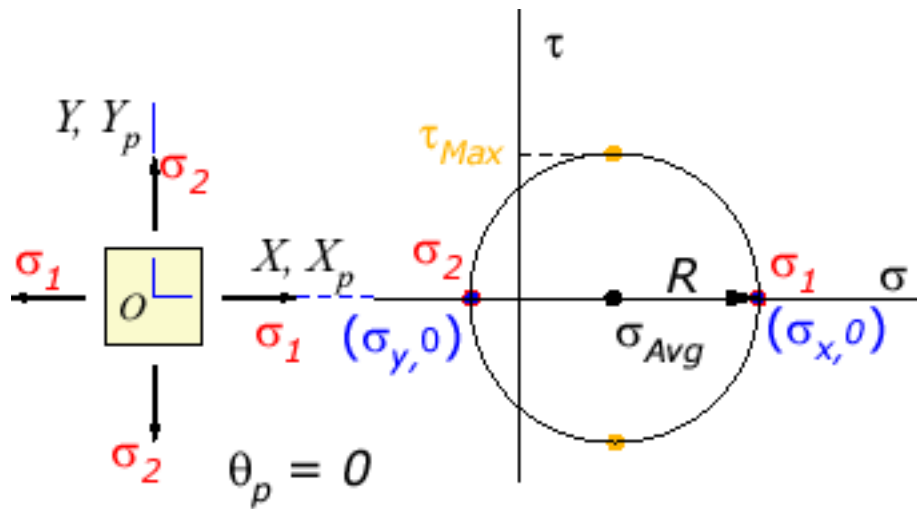
Case 4: $\tau_{xy} < 0$ and $\sigma_x < \sigma_y$

The principal axes are **clockwise** to the current axes (because $\tau_{xy} < 0$) and between 45° and 90° away (because $\sigma_x < \sigma_y$).



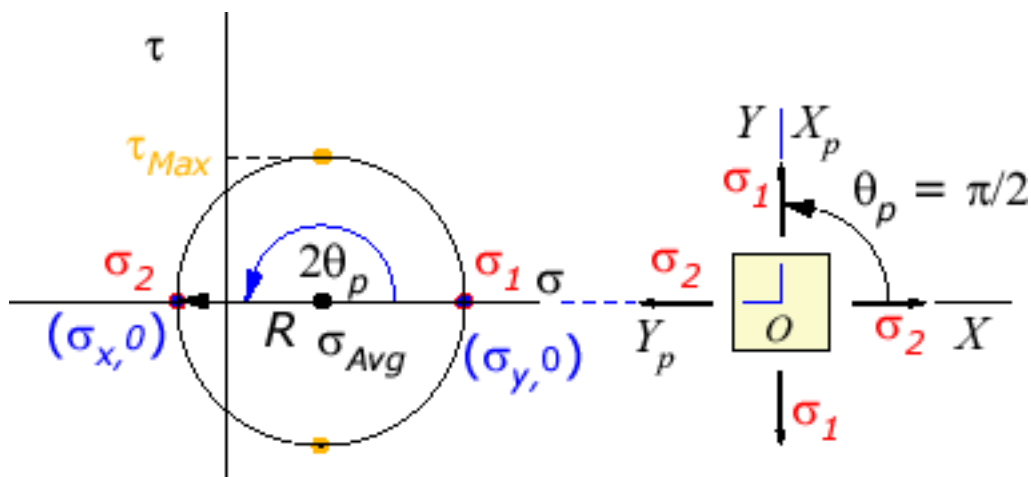
Case 5: $\tau_{xy} = 0$ and $\sigma_x > \sigma_y$

The principal axes are aligned with the current axes (because $\sigma_x > \sigma_y$ and $\tau_{xy} = 0$).



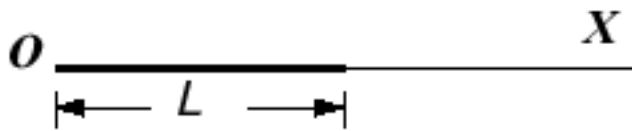
Case 6: $\tau_{xy} = 0$ and $\sigma_x < \sigma_y$

The principal axes are exactly 90° from the current axes (because $\sigma_x < \sigma_y$ and $\tau_{xy} = 0$).



Mechanics of Materials: Strain

Global 1D Strain



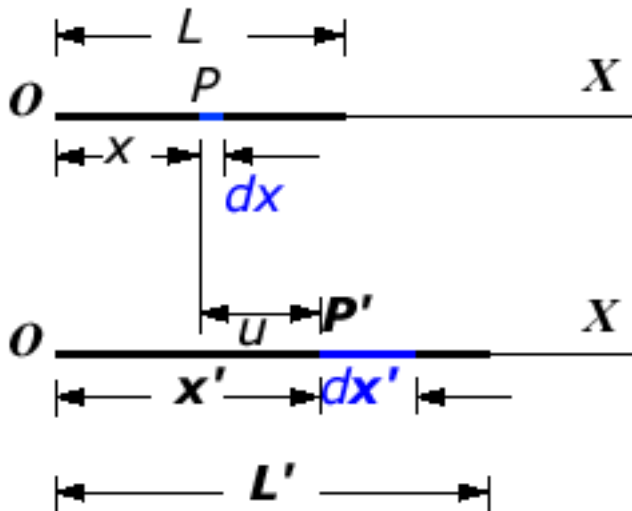
Consider a rod with initial length L which is stretched to a length L' . The strain measure ϵ , a dimensionless ratio, is defined as the ratio of elongation with respect to the original length,



$$\epsilon = \frac{L' - L}{L}$$

Infinitesimal 1D Strain

The above strain measure is defined in a global sense. The strain at each point may vary dramatically if the bar's elastic modulus or cross-sectional area changes. To track down the strain at each point, further refinement in the definition is needed.



Consider an arbitrary point in the bar P , which has a position vector x , and its infinitesimal neighbor dx . Point P shifts to P' , which has a position vector x' , after the stretch. In the meantime, the small "step" dx is stretched to dx' .

The strain at point p can be defined the same as in the global strain measure,

$$\epsilon = \frac{dx' - dx}{dx}$$

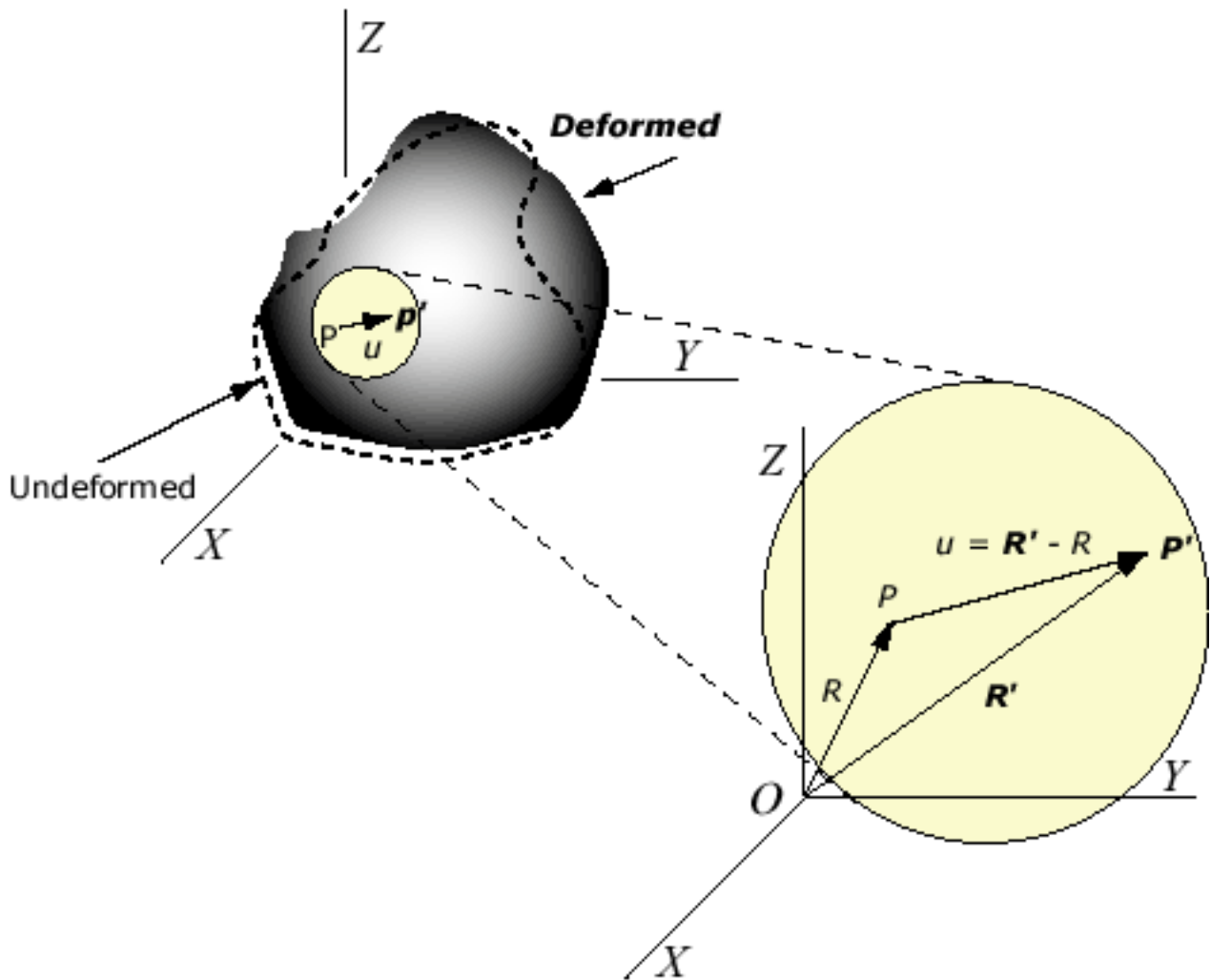
Since the displacement $u = x' - x$, the strain can

hence be rewritten as,

$$\epsilon = \frac{dx' - dx}{dx} = \frac{du}{dx}$$

General Definition of 3D Strain

As in the one dimensional strain derivation, suppose that point P in a body shifts to point P' after deformation.



The infinitesimal strain-displacement relationships can be summarized as,

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

where \mathbf{u} is the displacement vector, x is coordinate, and the two indices i and j can range over the three coordinates $\{1, 2, 3\}$ in three dimensional space.

Expanding the above equation for each coordinate direction gives,

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} & \varepsilon_{yz} &= \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) = \varepsilon_{zy} \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} & \varepsilon_{zx} &= \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \varepsilon_{xz} \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} & \varepsilon_{xy} &= \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = \varepsilon_{yx} \end{aligned}$$

where u , v , and w are the displacements in the x , y , and z directions respectively (i.e. they are the

components of \mathbf{u}).

3D Strain Matrix

There are a total of 6 strain measures. These 6 measures can be organized into a matrix (similar in form to the [3D stress matrix](#)), shown here,

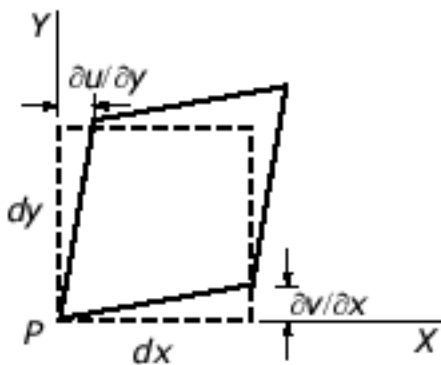
$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

Engineering Shear Strain

Focus on the strain ε_{xy} for a moment. The expression inside the parentheses can be rewritten as,

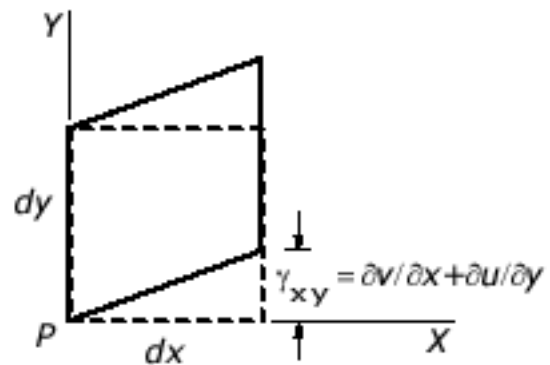
$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$. Called the **engineering shear strain**, γ_{xy} is a total measure of shear strain in the x - y plane. In contrast, the shear strain ε_{xy} is the average of the shear strain on the x face along the y direction, and on the y face along the x direction.



Shear strain tensor is the **average** of two strains, i.e.,

$$\varepsilon_{xy} = (\partial v / \partial x + \partial u / \partial y) / 2 = \varepsilon_{yx}$$



Engineer shear strain is the **total** shear strain, i.e.,

$$\gamma_{xy} = \partial v / \partial x + \partial u / \partial y$$

Engineering shear strain is commonly used in engineering reference books. However, please beware of the difference between shear strain and engineering shear strain, so as to avoid errors in mathematical manipulations.

Compatibility Conditions

In the strain-displacement relationships, there are six strain measures but only three independent displacements. That is, there are 6 unknowns for only 3 independent variables. As a result there exist 3 constraint, or compatibility, equations.

These compatibility conditions for infinitesimal strain referred to rectangular Cartesian coordinates are,

$$\begin{aligned} \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} &= 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} & \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} &= \frac{\partial}{\partial x} \left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right) \\ \frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} &= 2 \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} & \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right) \\ \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} &= 2 \frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} & \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} &= \frac{\partial}{\partial z} \left(\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z} \right) \end{aligned}$$

In two dimensional problems (e.g. [plane strain](#)), all z terms are set to zero. The compatibility equations reduce to,

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$$

Note that some references use engineering shear strain ($\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$) when referencing compatibility equations.

Plane Strain and Coordinate Transformations

Plane State of Strain

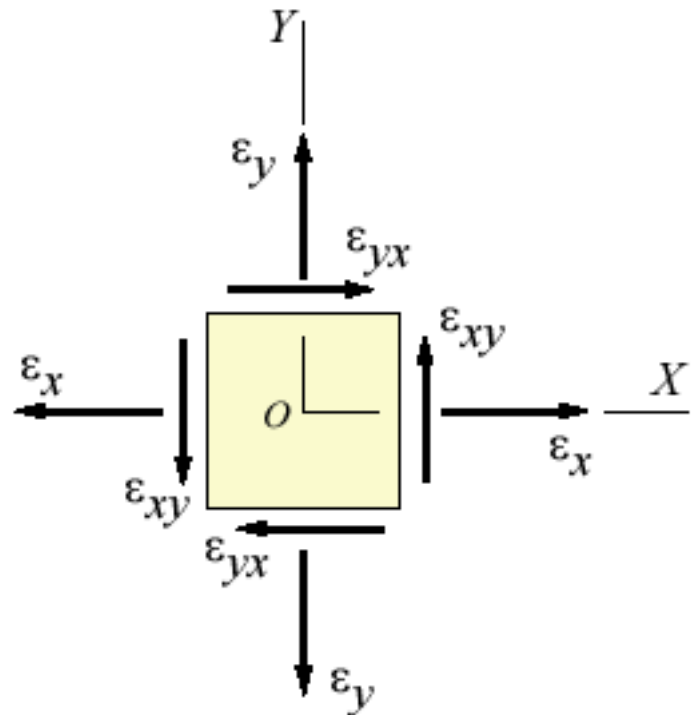
Some common engineering problems such as a dam subjected to water loading, a tunnel under external pressure, a pipe under internal pressure, and a cylindrical roller bearing compressed by force in a diametral plane, have significant strain only in a plane; that is, the strain in one direction is much less than the strain in the two other orthogonal directions. If small enough, the smallest strain can be ignored and the part is said to experience **plane strain**.

Assume that the negligible strain is oriented in the z -direction. To reduce the [3D strain matrix](#) to the 2D plane stress matrix, remove all components with z subscripts to get,

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_y \end{bmatrix}$$

where $\varepsilon_{xy} = \varepsilon_{yx}$ by definition.

The sign convention here is consistent with the sign convention used in [plane stress](#) analysis.

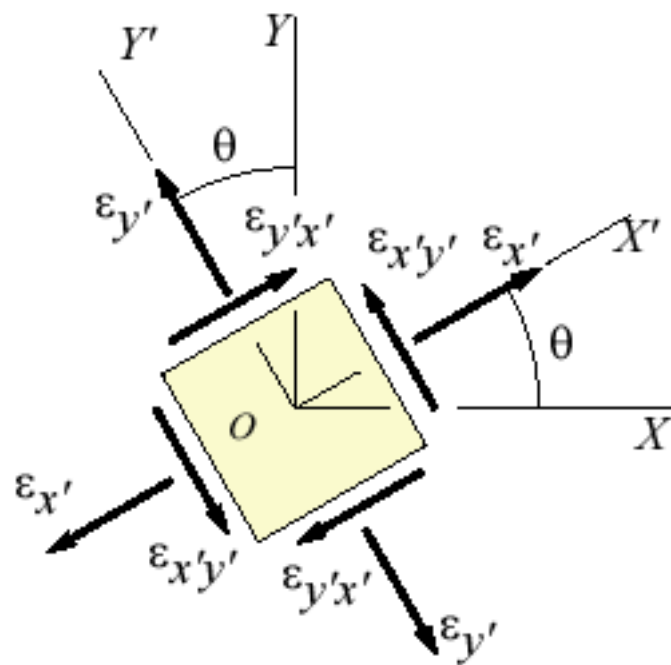


Coordinate Transformation

The transformation of strains with respect to the $\{x,y,z\}$ coordinates to the strains with respect to $\{x',y',z'\}$ is performed via the equations,

$$\begin{cases} \varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \varepsilon_{xy} \sin 2\theta \\ \varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta - \varepsilon_{xy} \sin 2\theta \\ \quad = \varepsilon_x + \varepsilon_y - \varepsilon_{x'} \\ \varepsilon_{x'y'} = -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \varepsilon_{xy} \cos 2\theta \end{cases}$$

The rotation between the two coordinate sets is shown here,



where θ is defined positive in the counterclockwise direction.

Principal Strain for the Case of Plane Strain

Principal Directions, Principal Strain

The normal strains (ϵ_x and ϵ_y) and the shear strain (ϵ_{xy}) vary smoothly with respect to the rotation angle θ , in accordance with the transformation equations given above. There exist a couple of particular angles where the strains take on special values.

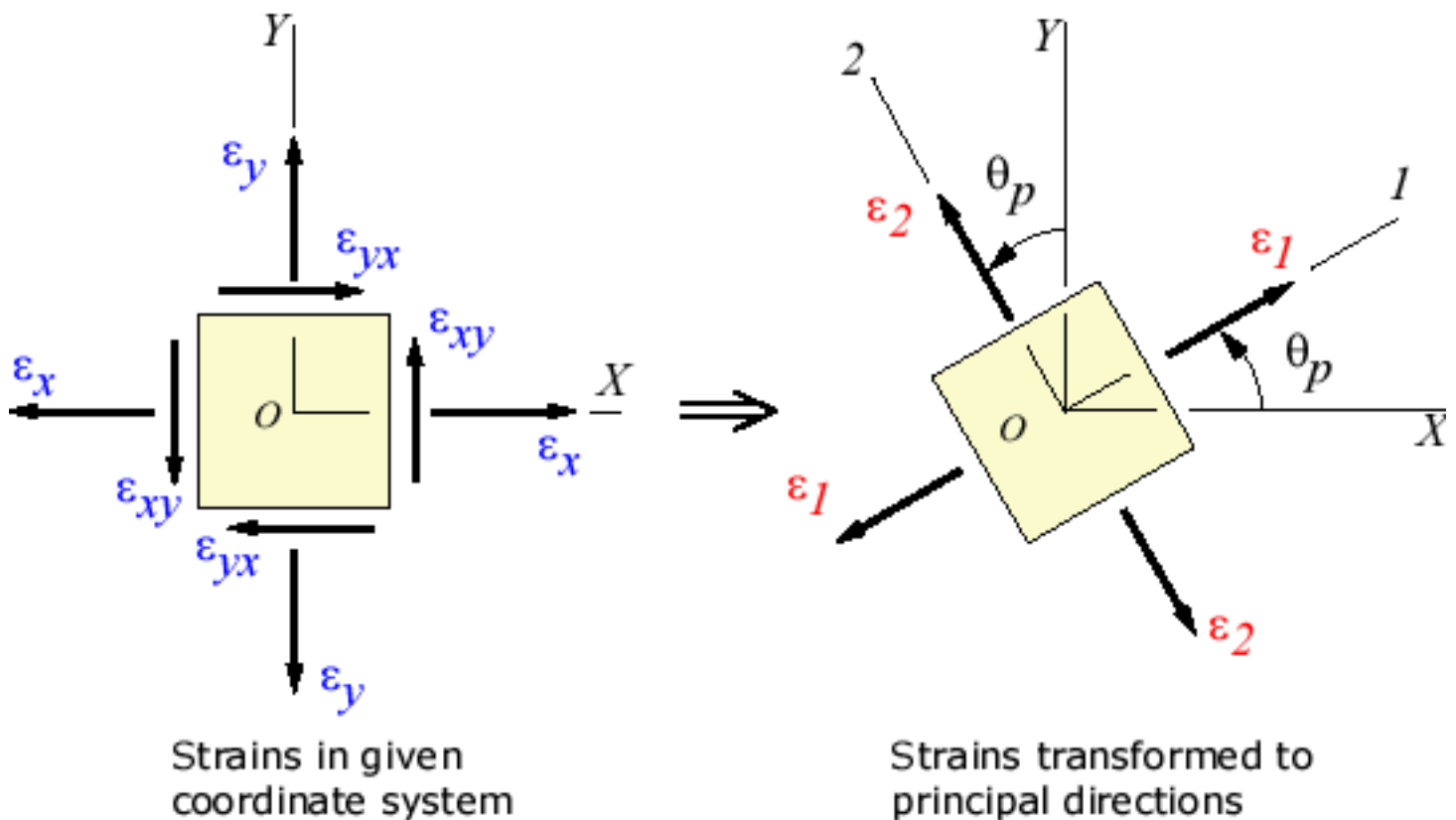
First, there exists an angle θ_p where the shear strain $\epsilon_{x'y'}$ vanishes. That angle is given by,

$$\tan 2\theta_p = \frac{2\epsilon_{xy}}{\epsilon_x - \epsilon_y}$$

This angle defines the *principal directions*. The associated *principal strains* are given by,

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \epsilon_{xy}^2}$$

The transformation to the principal directions with their principal strains can be illustrated as:



Maximum Shear Strain Direction

Another important angle, θ_s , is where the maximum shear strain occurs and is given by,

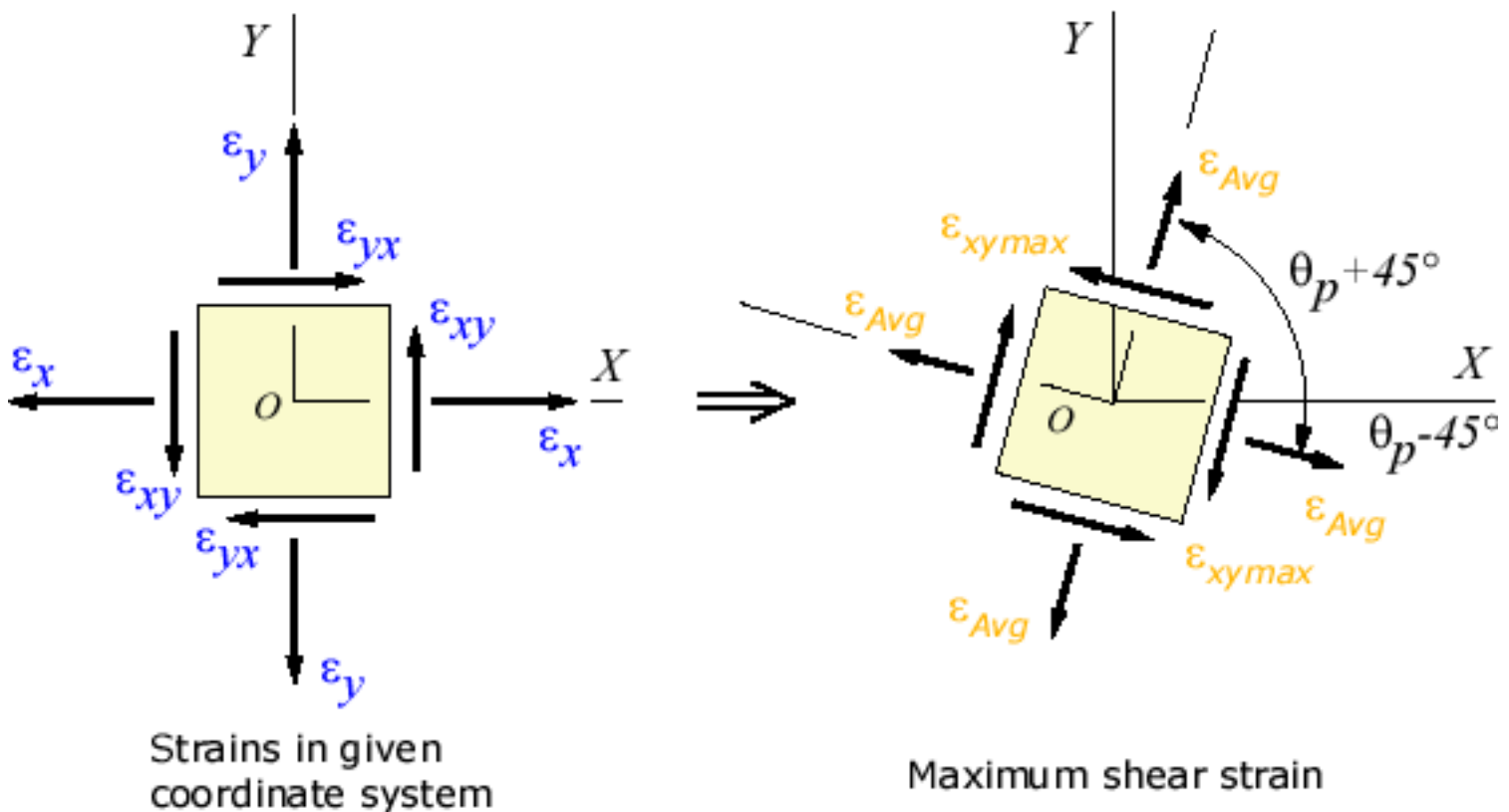
$$\tan 2\theta_s = -\frac{\varepsilon_x - \varepsilon_y}{2\varepsilon_{xy}}$$

$$\Rightarrow \theta_s = \theta_p \pm 45^\circ$$

The maximum shear strain is found to be one-half the difference between the two principal strains,

$$\varepsilon_{\max} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2} = \frac{\varepsilon_1 - \varepsilon_2}{2}$$

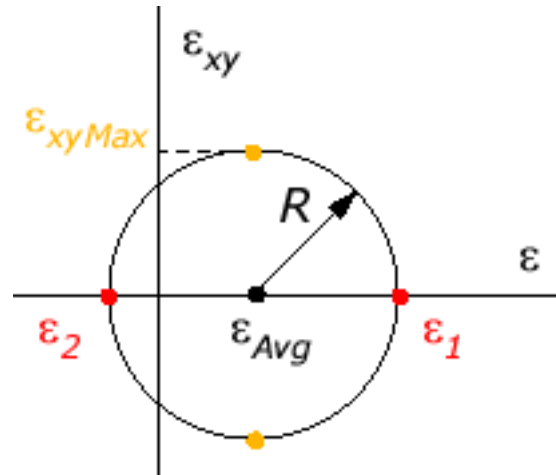
The transformation to the maximum shear strain direction can be illustrated as:



Mohr's Circle for Plane Strain

Mohr's Circle

Strains at a point in the body can be illustrated by Mohr's Circle. The idea and procedures are exactly the same as for [Mohr's Circle for plane stress](#).



The two principal strains are shown in **red**, and the maximum shear strain is shown in **orange**. Recall that the normal strains are equal to the principal strains when the element is aligned with the principal directions, and the shear strain is equal to the maximum shear strain when the element is rotated 45° away from the principal directions.

As the element is rotated away from the [principal](#) (or maximum strain) directions, the normal and shear strain components will always lie on Mohr's Circle.

Derivation of Mohr's Circle

To establish the Mohr's circle, we first recall the [strain transformation formulas](#) for plane strain,

$$\begin{cases} \varepsilon_{x'} - \frac{\varepsilon_x + \varepsilon_y}{2} = \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \varepsilon_{xy} \sin 2\theta \\ \varepsilon_{x'y'} = -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \varepsilon_{xy} \cos 2\theta \end{cases}$$

Using a [basic trigonometric relation](#) ($\cos^2 2\theta + \sin^2 2\theta = 1$) to combine the above two formulas we have,

$$\left(\varepsilon_{x'} - \frac{\varepsilon_x + \varepsilon_y}{2} \right)^2 + \varepsilon_{x'y'}^2 = \left(\frac{\varepsilon_x - \varepsilon_y}{2} \right)^2 + \varepsilon_{xy}^2$$

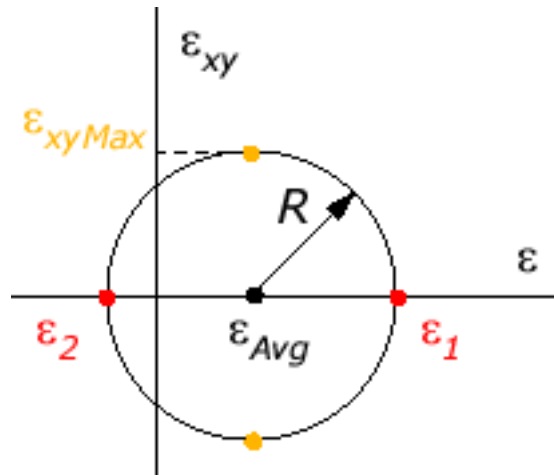
This equation is an equation for a circle. To make this more apparent, we can rewrite it as,

$$(\varepsilon_{x'} - \varepsilon_{Avg})^2 + \varepsilon_{x'y'}^2 = R^2$$

where,

$$\varepsilon_{Avg} = \frac{\varepsilon_x + \varepsilon_y}{2} \quad R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2}$$

The circle is centered at the average strain value ε_{Avg} , and has a radius R equal to the maximum shear strain, as shown in the figure below,



Related Topics

The procedure of drawing Mohr's Circle from a given strain state is discussed in the [Mohr's Circle usage](#) and [examples](#) pages.

The Mohr's Circle for [plane stress](#) can also be obtained from similar procedures.

Mohr's Circle Usage in Plane Strain

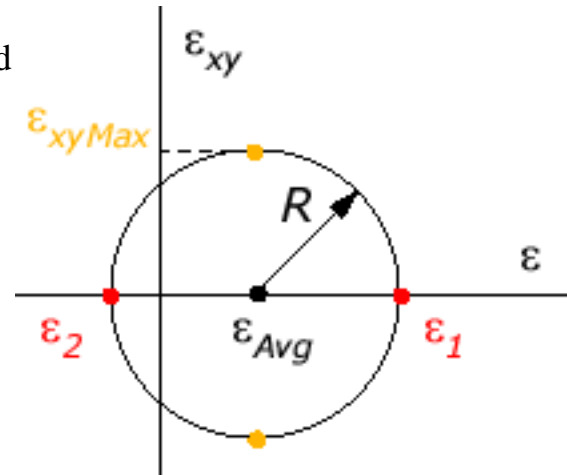
Principal Strains from Mohr's Circle

A chief benefit of Mohr's circle is that the [principal strains](#) ϵ_1 and ϵ_2 and the maximum shear strain ϵ_{xyMax} are obtained immediately after drawing the circle,

$$\begin{cases} \epsilon_{1,2} = \epsilon_{Avg} \pm R \\ \epsilon_{xyMax} = R \end{cases}$$

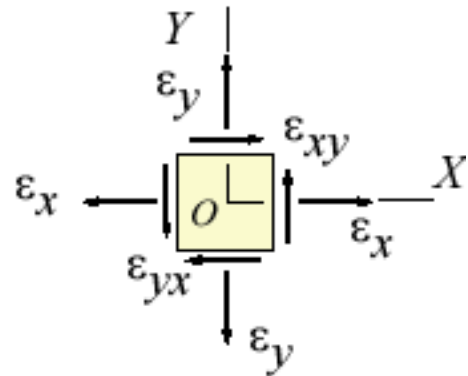
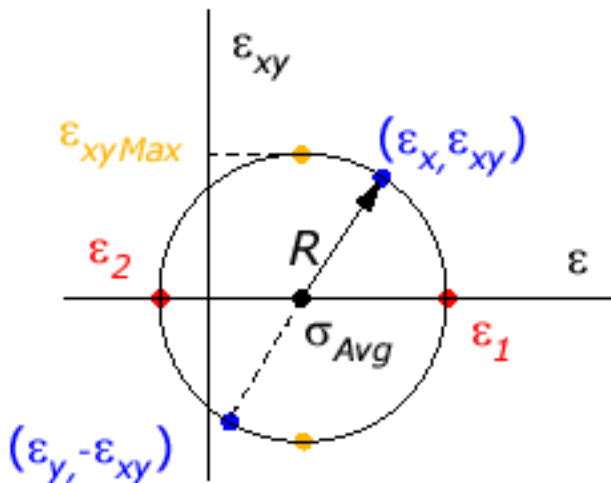
where,

$$\epsilon_{Avg} = \frac{\epsilon_x + \epsilon_y}{2} \quad R = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \epsilon_{xy}^2}$$



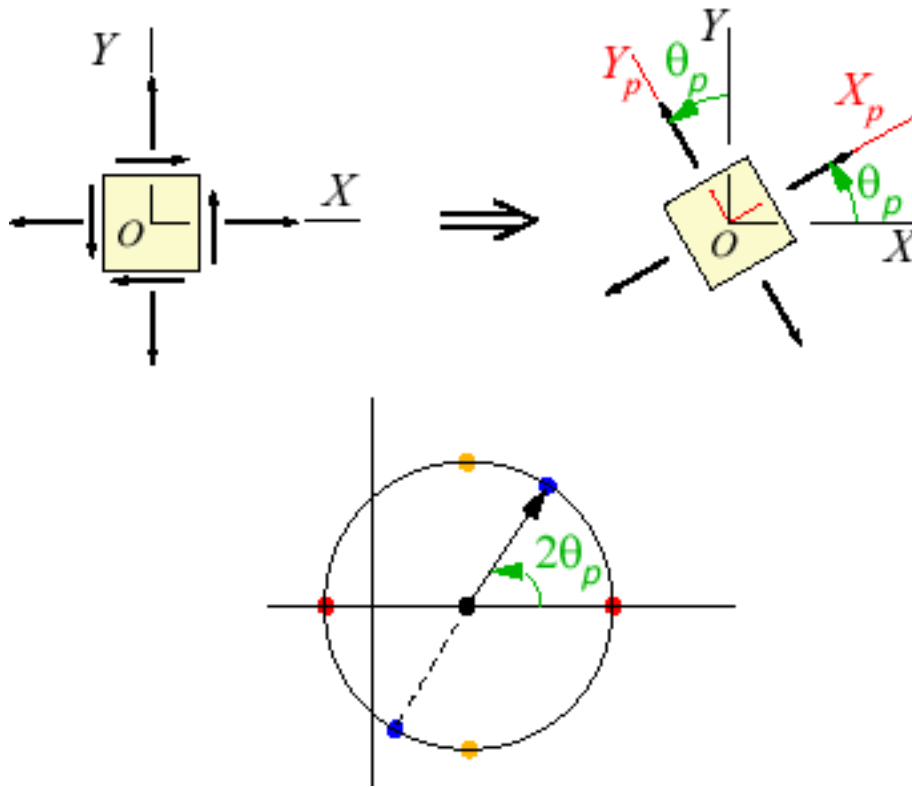
Principal Directions from Mohr's Circle

Mohr's Circle can be used to find the directions of the principal axes. To show this, first suppose that the normal and shear strains, ϵ_x , ϵ_y , and ϵ_{xy} , are obtained at a given point O in the body. They are expressed relative to the coordinates XY , as shown in the strain element at right below.



The Mohr's Circle for this general strain state is shown at left above. Note that it's centered at ϵ_{Avg} and has a radius R , and that the two points $(\epsilon_x, \epsilon_{xy})$ and $(\epsilon_y, -\epsilon_{xy})$ lie on opposite sides of the circle. The line connecting ϵ_x and ϵ_y will be defined as L_{xy} .

The **angle** between the current axes (X and Y) and the **principal axes** is defined as θ_p , and is equal to one half the angle between the line L_{xy} and the ϵ -axis as shown in the schematic below,



A set of six Mohr's Circles representing most strain state possibilities are presented on the [examples](#) page.

Also, principal directions can be computed by the [principal strain calculator](#).

Rotation Angle on Mohr's Circle

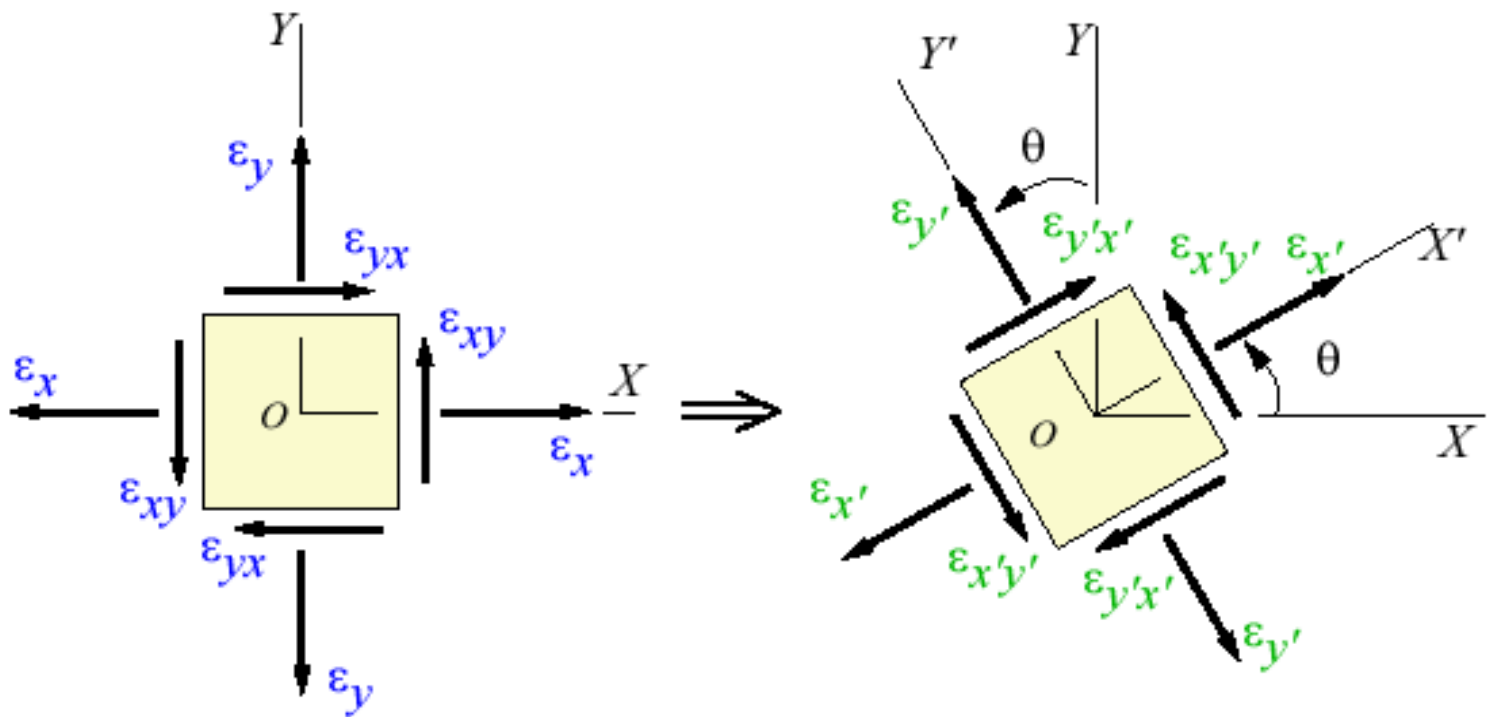
Note that the coordinate rotation angle θ_p is defined positive when starting at the XY coordinates and proceeding to the X_pY_p coordinates. In contrast, on the Mohr's Circle θ_p is defined positive starting on the principal strain line (i.e. the ϵ -axis) and proceeding to the XY strain line (i.e. line L_{xy}). The angle θ_p has the opposite sense between the two figures, because on one it starts on the XY coordinates, and on the other it starts on the principal coordinates.

Some books avoid the sign difference between θ_p on Mohr's Circle and θ_p on the stress element by locating $(\epsilon_x, -\epsilon_{xy})$ instead of $(\epsilon_x, \epsilon_{xy})$ on Mohr's Circle. This will switch the polarity of θ_p on the circle. Whatever method you choose, the bottom line is that an *opposite* sign is needed either in the interpretation or in the plotting to make Mohr's Circle physically meaningful.

Strain Transform by Mohr's Circle

Mohr's Circle can be used to transform strains from one coordinate set to another, similar that that described on the [plane strain](#) page.

Suppose that the normal and shear strains, ϵ_x , ϵ_y , and ϵ_{xy} , are obtained at a point O in the body, expressed with respect to the coordinates XY . We wish to find the strains expressed in the new coordinate set $X'Y'$, rotated an angle θ from XY , as shown below:

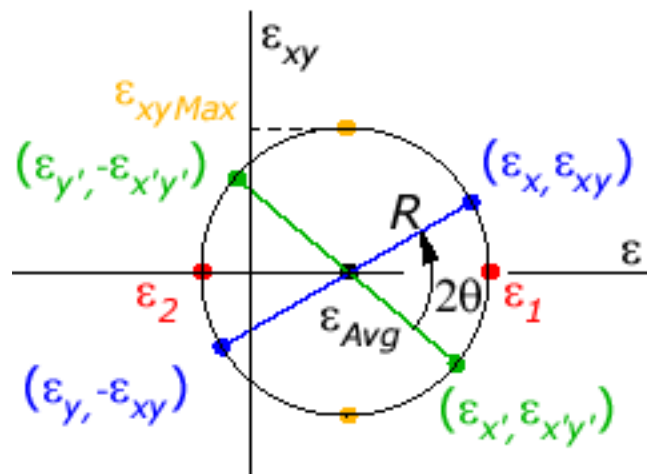


Strains at the given coordinate

Strains transformed to another coordinate

To do this we proceed as follows:

- Draw Mohr's circle for the **given strain state** (ϵ_x , ϵ_y , and ϵ_{xy} ; shown below).
- Draw the line L_{xy} across the circle from $(\epsilon_x, \epsilon_{xy})$ to $(\epsilon_y, -\epsilon_{xy})$.
- Rotate the line L_{xy} by $2*\theta$ (twice as much as the angle between XY and $X'Y'$) and in the *opposite* direction of θ .
- The **strains in the new coordinates** ($\epsilon_{x'}$, $\epsilon_{y'}$, and $\epsilon_{x'y'}$) are then read off the circle.

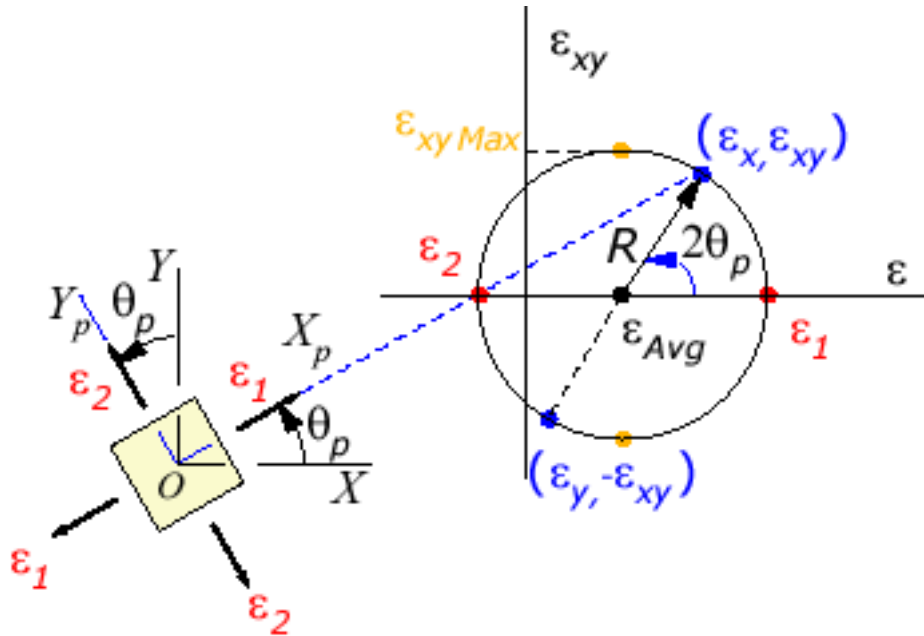


Strain transforms can be performed using eFunda's [strain transform calculator](#).

Examples of Mohr's Circles in Plane Strain

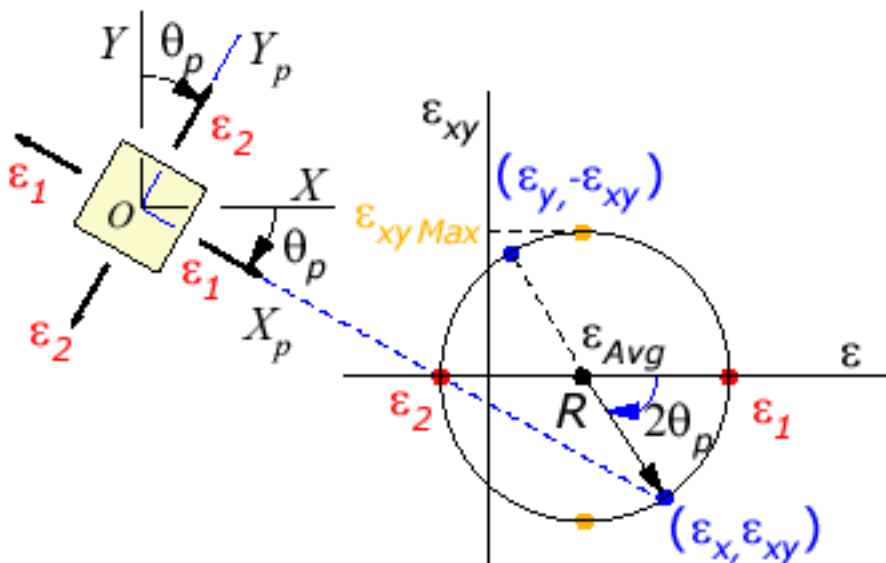
Case 1: $\epsilon_{xy} > 0$ and $\epsilon_x > \epsilon_y$

The principal axes are **counterclockwise** to the current axes (because $\epsilon_{xy} > 0$) and no more than 45° away (because $\epsilon_x > \epsilon_y$).



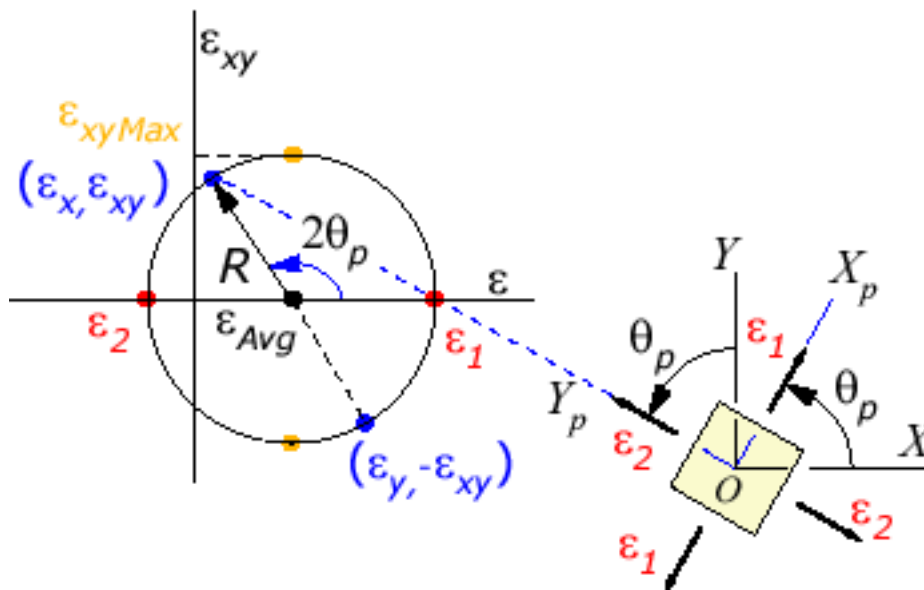
Case 2: $\epsilon_{xy} < 0$ and $\epsilon_x > \epsilon_y$

The principal axes are **clockwise** to the current axes (because $\epsilon_{xy} < 0$) and no more than 45° away (because $\epsilon_x > \epsilon_y$).



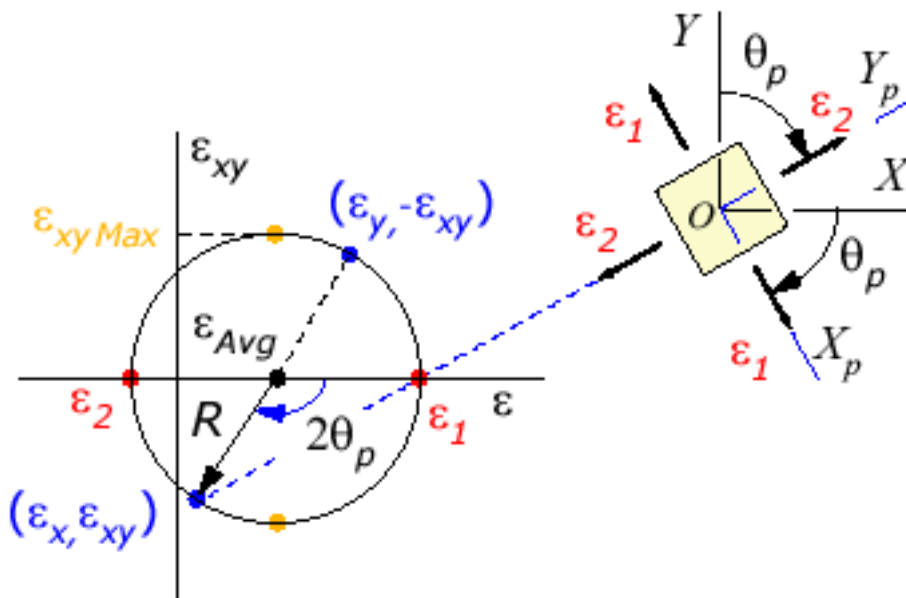
Case 3: $\epsilon_{xy} > 0$ and $\epsilon_x < \epsilon_y$

The principal axes are **counterclockwise** to the current axes (because $\epsilon_{xy} > 0$) and between 45° and 90° away (because $\epsilon_x < \epsilon_y$).



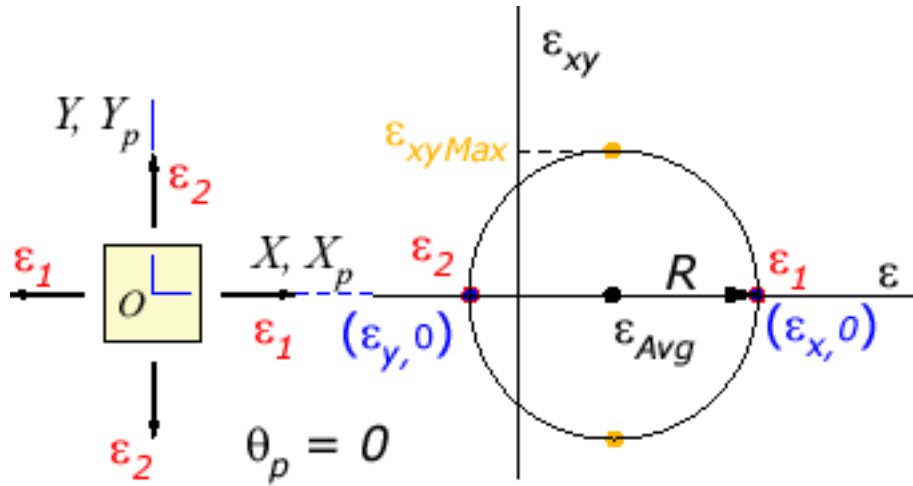
Case 4: $\epsilon_{xy} < 0$ and $\epsilon_x < \epsilon_y$

The principal axes are **clockwise** to the current axes (because $\epsilon_{xy} < 0$) and between 45° and 90° away (because $\epsilon_x < \epsilon_y$).



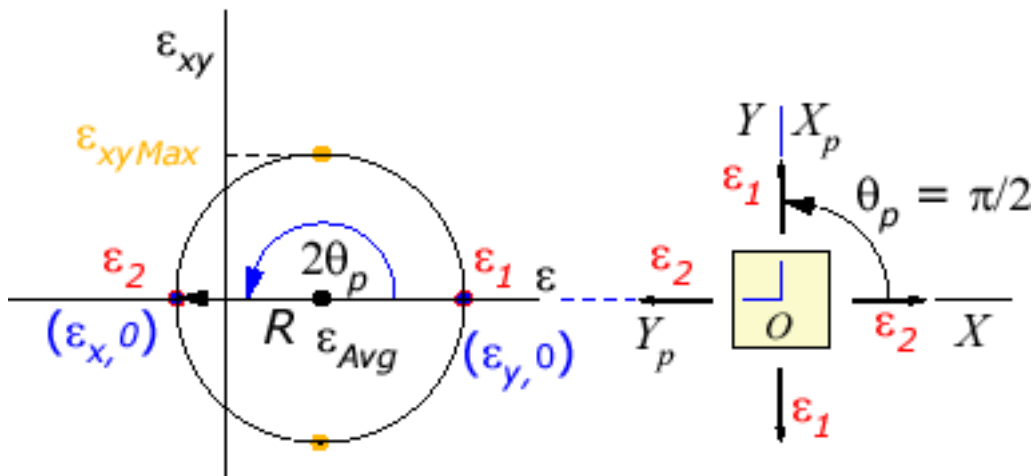
Case 5: $\epsilon_{xy} = 0$ and $\epsilon_x > \epsilon_y$

The principal axes are aligned with the current axes (because $\epsilon_x > \epsilon_y$ and $\epsilon_{xy} = 0$).



Case 6: $\epsilon_{xy} = 0$ and $\epsilon_x < \epsilon_y$

The principal axes are exactly 90° from the current axes (because $\epsilon_x < \epsilon_y$ and $\epsilon_{xy} = 0$).



Mechanics of Materials: Hooke's Law

One-dimensional Hooke's Law

Robert Hooke, who in 1676 stated,

The power (*sic.*) of any springy body is in the same proportion with the extension.

announced the birth of elasticity. Hooke's statement expressed mathematically is,

$$F = k \cdot u$$

where F is the applied force (and not the power, as Hooke mistakenly suggested), u is the deformation of the elastic body subjected to the force F , and k is the spring constant (i.e. the ratio of previous two parameters).

Generalized Hooke's Law (Anisotropic Form)

Cauchy generalized Hooke's law to three dimensional elastic bodies and stated that the 6 components of stress are linearly related to the 6 components of strain.

The stress-strain relationship written in matrix form, where the 6 components of [stress](#) and [strain](#) are organized into column vectors, is,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}, \quad \varepsilon = \mathbf{S} \cdot \sigma$$

or,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix}, \quad \sigma = \mathbf{C} \cdot \varepsilon$$

where \mathbf{C} is the **stiffness matrix**, \mathbf{S} is the **compliance matrix**, and $\mathbf{S} = \mathbf{C}^{-1}$.

In general, stress-strain relationships such as these are known as **constitutive relations**.

In general, there are 36 stiffness matrix components. However, it can be shown that conservative materials possess a strain energy density function and as a result, the stiffness and compliance matrices are symmetric. Therefore, only 21 stiffness components are actually independent in Hooke's law. The vast majority of engineering materials are conservative.

Please note that the **stiffness** matrix is traditionally represented by the symbol **C**, while **S** is reserved for the **compliance** matrix. This convention may seem backwards, but perception is not always reality. For instance, Americans hardly ever use their feet to play (American) football.

Hooke's Law for Orthotropic Materials

Orthotropic Definition

Some engineering materials, including certain piezoelectric materials (e.g. [Rochelle salt](#)) and 2-ply fiber-reinforced composites, are **orthotropic**.

By definition, an orthotropic material has at least 2 orthogonal planes of symmetry, where material properties are independent of direction within each plane. Such materials require 9 independent variables (i.e. elastic constants) in their constitutive matrices.

In contrast, a material without any planes of symmetry is fully [anisotropic](#) and requires 21 elastic constants, whereas a material with an infinite number of symmetry planes (i.e. every plane is a plane of symmetry) is [isotropic](#), and requires only 2 elastic constants.

Hooke's Law in Compliance Form

By convention, the 9 elastic constants in orthotropic constitutive equations are comprised of 3 Young's moduli E_x , E_y , E_z , the 3 Poisson's ratios ν_{yz} , ν_{zx} , ν_{xy} , and the 3 shear moduli G_{yz} , G_{zx} , G_{xy} .

The **compliance matrix** takes the form,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{zx}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}$$

where $\frac{\nu_{yz}}{E_y} = \frac{\nu_{zy}}{E_z}$, $\frac{\nu_{zx}}{E_z} = \frac{\nu_{xz}}{E_x}$, $\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y}$.

Note that, in orthotropic materials, there is no interaction between the normal stresses σ_x , σ_y , σ_z and the shear strains ϵ_{yz} , ϵ_{zx} , ϵ_{xy}

The factor 2 multiplying the shear moduli in the compliance matrix results from the difference between shear strain and [engineering shear strain](#), where $\gamma_{xy} = \epsilon_{xy} + \epsilon_{yx} = 2\epsilon_{xy}$, etc.

Hooke's Law in Stiffness Form

The **stiffness matrix** for orthotropic materials, found from the inverse of the compliance matrix, is given by,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1 - \nu_{yz}\nu_{zy}}{E_y E_z \Delta} & \frac{\nu_{yx} + \nu_{zx}\nu_{yz}}{E_y E_z \Delta} & \frac{\nu_{zx} + \nu_{yx}\nu_{zy}}{E_y E_z \Delta} & 0 & 0 & 0 \\ \frac{\nu_{xy} + \nu_{xz}\nu_{zx}}{E_z E_x \Delta} & \frac{1 - \nu_{zx}\nu_{xz}}{E_z E_x \Delta} & \frac{\nu_{zy} + \nu_{zx}\nu_{xy}}{E_z E_x \Delta} & 0 & 0 & 0 \\ \frac{\nu_{xz} + \nu_{xy}\nu_{yz}}{E_x E_y \Delta} & \frac{\nu_{zy} + \nu_{xz}\nu_{yz}}{E_x E_y \Delta} & \frac{1 - \nu_{xy}\nu_{yx}}{E_x E_y \Delta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G_{yz} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G_{zx} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G_{xy} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{zx} \\ \epsilon_{xy} \end{bmatrix}$$

where,

$$\Delta = \frac{1 - \nu_{xy}\nu_{yx} - \nu_{yz}\nu_{zy} - \nu_{zx}\nu_{xz} - \nu_{xy}\nu_{yz}\nu_{zx}}{E_x E_y E_z}$$

The fact that the stiffness matrix is symmetric requires that the following statements hold,

$$\begin{cases} \frac{\nu_{yx} + \nu_{zx}\nu_{yz}}{E_y E_z \Delta} = \frac{\nu_{xy} + \nu_{xz}\nu_{zx}}{E_z E_x \Delta} \\ \frac{\nu_{zy} + \nu_{zx}\nu_{xy}}{E_z E_x \Delta} = \frac{\nu_{zy} + \nu_{xz}\nu_{yz}}{E_x E_y \Delta} \\ \frac{\nu_{zx} + \nu_{yx}\nu_{zy}}{E_y E_z \Delta} = \frac{\nu_{xz} + \nu_{xy}\nu_{yz}}{E_x E_y \Delta} \end{cases}$$

The factor of 2 multiplying the shear moduli in the stiffness matrix results from the difference between shear strain and [engineering shear strain](#), where $\gamma_{xy} = \epsilon_{xy} + \epsilon_{yx} = 2\epsilon_{xy}$, etc.

Hooke's Law for Transversely Isotropic Materials

Transverse Isotropic Definition

A special class of [orthotropic](#) materials are those that have the same properties in one plane (e.g. the x - y plane) and different properties in the direction normal to this plane (e.g. the z -axis). Such materials are called **transverse isotropic**, and they are described by 5 independent elastic constants, instead of 9 for fully orthotropic.

Examples of transversely isotropic materials include some piezoelectric materials (e.g. [PZT-4](#), [barium titanate](#)) and fiber-reinforced composites where all fibers are in parallel.

Hooke's Law in Compliance Form

By convention, the 5 elastic constants in transverse isotropic constitutive equations are the Young's modulus and poisson ratio in the x - y symmetry plane, E_p and ν_p , the Young's modulus and poisson ratio in the z -direction, E_{pz} and ν_{pz} , and the shear modulus in the z -direction G_{zp} .

The **compliance matrix** takes the form,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_p} & -\frac{\nu_p}{E_p} & -\frac{\nu_{zp}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_p}{E_p} & \frac{1}{E_p} & -\frac{\nu_{zp}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{pz}}{E_p} & -\frac{\nu_{pz}}{E_p} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{zp}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{zp}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+\nu_p}{E_p} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}$$

where $\frac{\nu_{pz}}{E_p} = \frac{\nu_{zp}}{E_z}$.

The factor 2 multiplying the shear moduli in the compliance matrix results from the difference between shear strain and [engineering shear strain](#), where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$, etc.

Hooke's Law in Stiffness Form

The **stiffness matrix** for transverse isotropic materials, found from the inverse of the compliance matrix, is given by,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1 - \nu_{pz}\nu_{zp}}{E_p E_z \Delta} & \frac{\nu_p + \nu_{zp}\nu_{pz}}{E_p E_z \Delta} & \frac{\nu_{zp} + \nu_p\nu_{pz}}{E_p E_z \Delta} & 0 & 0 & 0 \\ \frac{\nu_p + \nu_{pz}\nu_{zp}}{E_z E_p \Delta} & \frac{1 - \nu_{zp}\nu_{pz}}{E_z E_p \Delta} & \frac{\nu_{zp} + \nu_{zp}\nu_p}{E_z E_p \Delta} & 0 & 0 & 0 \\ \frac{\nu_{pz} + \nu_p\nu_{pz}}{E_p^2 \Delta} & \frac{\nu_{zp} + \nu_{pz}^2}{E_p^2 \Delta} & \frac{1 - \nu_p^2}{E_p^2 \Delta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G_{zp} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G_{zp} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{E_p}{1 + \nu_p} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix}$$

where,

$$\Delta = \frac{1 - \nu_p^2 - 2\nu_{pz}\nu_{zp} - \nu_p\nu_{pz}\nu_{zp}}{E_p^2 E_z}$$

The fact that the stiffness matrix is symmetric requires that the following statements hold,

$$\begin{cases} \frac{\nu_p + \nu_{zp}\nu_{pz}}{E_p E_z \Delta} = \frac{\nu_p + \nu_{pz}\nu_{zp}}{E_z E_p \Delta} \\ \frac{\nu_{zp} + \nu_{zp}\nu_p}{E_z E_p \Delta} = \frac{\nu_{zp} + \nu_{pz}^2}{E_p^2 \Delta} \\ \frac{\nu_{zp} + \nu_p\nu_{zp}}{E_p E_z \Delta} = \frac{\nu_{pz} + \nu_p\nu_{pz}}{E_p^2 \Delta} \end{cases}$$

The factor of 2 multiplying the shear moduli in the stiffness matrix results from the difference

between shear strain and [engineering shear](#) strain, where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$, etc.

Hooke's Law for Isotropic Materials

Isotropic Definition

Most metallic alloys and thermoset polymers are considered **isotropic**, where by definition the material properties are independent of direction. Such materials have only 2 independent variables (i.e. elastic constants) in their stiffness and compliance matrices, as opposed to the 21 elastic constants in the general [anisotropic](#) case.

The two elastic constants are usually expressed as the [Young's modulus](#) E and the [Poisson's ratio](#) ν . However, the alternative elastic constants K ([bulk modulus](#)) and/or G ([shear modulus](#)) can also be used. For isotropic materials, G and K can be found from E and ν by a set of [equations](#), and vice-versa.

Hooke's Law in Compliance Form

Hooke's law for isotropic materials in **compliance matrix** form is given by,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}$$

Hooke's Law in Stiffness Form

The **stiffness matrix** is equal to the inverse of the compliance matrix, and is given by,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix}$$

Visit the [elastic constant calculator](#) to see the interplay amongst the 4 elastic constants (E , ν , G , K).

Hooke's Law for Plane Stress

Hooke's Law for Plane Stress

For the simplification of [plane stress](#), where the stresses in the z direction are considered to be negligible, $\sigma_{zz} = \sigma_{yz} = \sigma_{xz} = 0$, the stress-strain compliance relationship for an [isotropic](#) material becomes,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ 0 \\ 0 \\ 0 \\ \sigma_{xy} \end{bmatrix}$$

The three zero'd stress entries in the stress vector indicate that we can ignore their associated columns in the compliance matrix (i.e. columns 3, 4, and 5). If we also ignore the rows associated with the strain components with z -subscripts, the **compliance matrix** reduces to a simple 3x3 matrix,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

The **stiffness matrix** for plane stress is found by inverting the plane stress compliance matrix, and is given by,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}$$

Note that the stiffness matrix for plane stress is **NOT** found by removing columns and rows from the general [isotropic stiffness matrix](#).

Plane Stress Hooke's Law via Engineering Strain

Some reference books incorporate the shear modulus G and the [engineering shear strain](#) γ_{xy} , related to the shear strain ε_{xy} via,

$$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$$

The stress-strain **compliance matrix** using G and γ_{xy} are,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

The **stiffness matrix** is,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

The shear modulus G is [related](#) to E and ν via,

$$G = \frac{E}{2(1+\nu)}$$

Hooke's Law for Plane Strain

Hooke's Law for Plane Strain

For the case of [plane strain](#), where the strains in the z direction are considered to be negligible, $\varepsilon_{zz} = \varepsilon_{yz} = \varepsilon_{xz} = 0$, the stress-strain stiffness relationship for an [isotropic](#) material becomes,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 0 \\ 0 \\ 0 \\ \varepsilon_{xy} \end{bmatrix}$$

The three zero'd strain entries in the strain vector indicate that we can ignore their associated columns in the stiffness matrix (i.e. columns 3, 4, and 5). If we also ignore the rows associated with the stress components with z -subscripts, the **stiffness matrix** reduces to a simple 3x3 matrix,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

The **compliance matrix** for plane stress is found by inverting the plane stress stiffness matrix, and is given by,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

Note that the compliance matrix for plane stress is **NOT** found by removing columns and rows from the general [isotropic compliance matrix](#).

Plane Strain Hooke's Law via Engineering Strain

The stress-strain **stiffness matrix** expressed using the shear modulus G and the engineering shear strain $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$ is,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 \\ \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

The **compliance matrix** is,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1-\nu^2}{E} & -\frac{\nu(1+\nu)}{E} & 0 \\ -\frac{\nu(1+\nu)}{E} & \frac{1-\nu^2}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

The shear modulus G is [related](#) to E and ν via,

$$G = \frac{E}{2(1+\nu)}$$

Finding Young's Modulus and Poisson's Ratio

Young's Modulus from Uniaxial Tension

When a specimen made from an [isotropic](#) material is subjected to uniaxial tension, say in the x direction, σ_{xx} is the only non-zero stress. The strains in the specimen are obtained by,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The modulus of elasticity in tension, also known as **Young's modulus** E , is the ratio of stress to strain on the loading plane along the loading direction,

$$E = \frac{\sigma_{xx}}{\varepsilon_{xx}}$$

Common sense (and the 2nd Law of Thermodynamics) indicates that a material under uniaxial tension must elongate in length. Therefore the Young's modulus E is required to be non-negative for all materials,

$$E > 0$$

Poisson's Ratio from Uniaxial Tension

A rod-like specimen subjected to uniaxial tension will exhibit some shrinkage in the lateral direction for most materials. The ratio of lateral strain and axial strain is defined as **Poisson's ratio** ν ,

$$\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}}$$

The Poisson ratio for most metals falls between 0.25 to 0.35. Rubber has a Poisson ratio close to 0.5 and is therefore almost incompressible. Theoretical materials with a Poisson ratio of **exactly 0.5** are truly **incompressible**, since the sum of all their strains leads to a zero volume change. Cork, on the other hand, has a Poisson ratio close to zero. This makes cork function well as a bottle stopper, since an axially-loaded cork will not swell laterally to resist bottle insertion.

The Poisson's ratio is bounded by two theoretical limits: it must be [greater than -1](#), and [less than or equal to 0.5](#),

$$-1 < \nu \leq \frac{1}{2}$$

The [proof](#) for this stems from the fact that E , G , and K are all positive and mutually dependent. However, it is rare to encounter engineering materials with negative Poisson ratios. Most materials will fall in the range,

$$0 \leq \nu \leq \frac{1}{2}$$

Finding the Shear Modulus and the Bulk Modulus

Shear Modulus from Pure Shear

When a specimen made from an [isotropic](#) material is subjected to pure shear, for instance, a cylindrical bar under torsion in the xy sense, σ_{xy} is the only non-zero stress. The strains in the specimen are obtained by,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sigma_{xy} \end{bmatrix}$$

The **shear modulus** G , is defined as the ratio of shear stress to [engineering shear strain](#) on the loading plane,

$$\begin{aligned} G &= \frac{\sigma_{xy}}{\varepsilon_{xy} + \varepsilon_{yx}} = \frac{\sigma_{xy}}{2\varepsilon_{xy}} = \frac{\sigma_{xy}}{\gamma_{xy}} \\ &= \frac{E}{2(1+\nu)} \end{aligned}$$

where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$.

The shear modulus G is also known as the rigidity modulus, and is equivalent to the 2nd Lamé constant μ mentioned in books on continuum theory.

Common sense and the 2nd Law of Thermodynamics require that a positive shear stress leads to a positive shear strain. Therefore, the shear modulus G is required to be nonnegative for all materials,

$$G > 0$$

Since both G and the elastic modulus E are required to be positive, the quantity in the denominator of G must also be positive. This requirement places a **lower bound restriction on the range for Poisson's ratio**,

$$\nu > -1$$

Bulk Modulus from Hydrostatic Pressure

When an [isotropic](#) material specimen is subjected to hydrostatic pressure σ , all shear stress will be zero and the normal stress will be uniform, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma$. The strains in the specimen are given by,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma \\ \sigma \\ \sigma \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In response to the hydrostatic load, the specimen will change its volume. Its resistance to do so is quantified as the **bulk modulus K** , also known as the modulus of compression. Technically, K is defined as the ratio of hydrostatic pressure to the [relative volume change](#) (which is related to the direct strains),

$$\begin{aligned} K &= \frac{\sigma}{\Delta V/V} = \frac{\sigma}{\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}} \\ &= \frac{E}{3(1-2\nu)} \end{aligned}$$

Common sense and the 2nd Law of Thermodynamics require that a positive hydrostatic load leads to a positive volume change. Therefore, the bulk modulus K is required to be nonnegative for all materials,

$$K > 0$$

Since both K and the elastic modulus E are required to be positive, the following requirement is placed on the **upper bound of Poisson's ratio** by the denominator of K ,

$$\nu < 1/2$$

Relation Between Relative Volume Change and Strain

For simplicity, consider a rectangular block of material with dimensions a_0 , b_0 , and c_0 . Its volume V_0 is given by,

$$V_0 = a_0 b_0 c_0$$

When the block is loaded by stress, its volume will change since each dimension now includes a direct strain measure. To calculate the volume when loaded V_f we multiply the new dimensions of the block,

$$\begin{aligned} V_f &= a_f b_f c_f = [a_0 (1 + \varepsilon_{xx})] [b_0 (1 + \varepsilon_{yy})] [c_0 (1 + \varepsilon_{zz})] \\ &= V_0 (1 + \varepsilon_{xx}) (1 + \varepsilon_{yy}) (1 + \varepsilon_{zz}) \\ &= V_0 (1 + \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx} + \varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz}) \\ &\approx V_0 (1 + \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) \end{aligned}$$

Products of strain measures will be much smaller than individual strain measures when the overall strain in the block is small (i.e. linear strain theory). Therefore, we were able to drop the strain products in the equation above.

The relative change in volume is found by dividing the volume difference by the initial volume,

$$\frac{\Delta V}{V_0} = \frac{V_f - V_0}{V_0} \approx \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

Hence, the relative volume change (for small strains) is equal to the sum of the 3 direct strains.

Failure Criteria

Stress-Based Criteria

The purpose of **failure criteria** is to predict or estimate the failure/yield of machine parts and structural members.

A considerable number of theories have been proposed. However, only the most common and well-tested theories applicable to [isotropic](#) materials are discussed here. These theories, dependent on the nature of the material in question (i.e. brittle or ductile), are listed in the following table:

Material Type	Failure Theories
Ductile	Maximum shear stress criterion , von Mises criterion
Brittle	Maximum normal stress criterion , Mohr's theory

All four criteria are presented in terms of [principal stresses](#). Therefore, all stresses should be [transformed](#) to the principal stresses before applying these failure criteria.

- Note:
- Whether a material is *brittle* or *ductile* could be a subjective guess, and often depends on temperature, strain levels, and other environmental conditions. However, a 5% *elongation* criterion at break is a reasonable dividing line. Materials with a larger elongation can be considered ductile and those with a lower value brittle. Another distinction is a brittle material's compression strength is usually significantly larger than its tensile strength.
 - All popular failure criteria rely on only a handful of basic tests (such as uniaxial tensile and/or compression strength), even though most machine parts and structural members are typically subjected to multi-axial loading. This disparity is usually driven by cost, since complete multi-axial failure testing requires extensive, complicated, and expensive tests.

Non Stress-Based Criteria

The success of all machine parts and structural members are not necessarily determined by their strength. Whether a part succeeds or fails may depend on other factors, such as stiffness, vibrational characteristics, fatigue resistance, and/or creep resistance.

For example, the automobile industry has endeavored many years to increase the rigidity of passenger cages and install additional safety equipment. The bicycle industry continues to decrease the weight and increase the stiffness of bicycles to enhance their performance.

In civil engineering, a patio deck only needs to be strong enough to carry the weight of several people. However, a design based on the "strong enough" precept will often result a bouncy deck that most people will find objectionable. Rather, the *stiffness* of the deck determines the success of the design.

Many factors, in addition to stress, may contribute to the design requirements of a part. Together, these requirements are intended to increase the sense of security, safety, and quality of service of the part.

Failure Criteria for Ductile Materials

Maximum Shear Stress Criterion

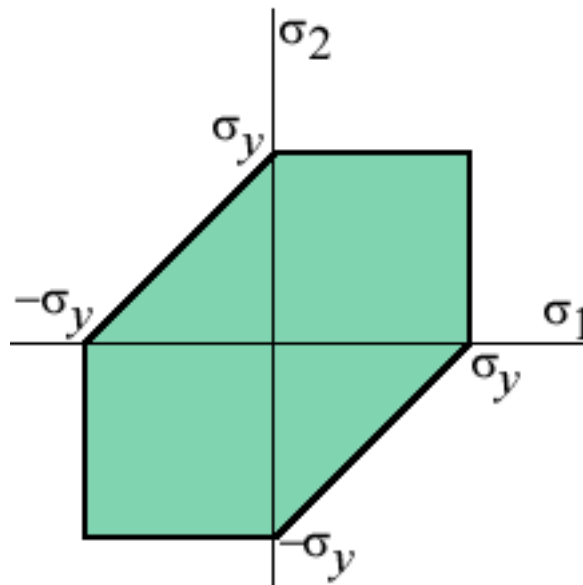
The maximum shear stress criterion, also known as Tresca's or Guest's criterion, is often used to predict the yielding of ductile materials.

Yield in ductile materials is usually caused by the *slippage* of crystal planes along the maximum shear stress surface. Therefore, a given point in the body is considered safe as long as the maximum shear stress at that point is under the yield shear stress σ_y obtained from a uniaxial tensile test.

With respect to 2D stress, the maximum shear stress is related to the difference in the two [principal stresses](#) (see [Mohr's Circle](#)). Therefore, the criterion requires the principal stress difference, along with the principal stresses themselves, to be less than the yield shear stress,

$$|\sigma_1| \leq \sigma_y, \quad |\sigma_2| \leq \sigma_y, \quad \text{and} \quad |\sigma_1 - \sigma_2| \leq \sigma_y$$

Graphically, the maximum shear stress criterion requires that the two principal stresses be within the green zone indicated below,



Von Mises Criterion

The von Mises Criterion (1913), also known as the maximum distortion energy criterion, octahedral shear stress theory, or Maxwell-Huber-Hencky-von Mises theory, is often used to estimate the yield of ductile materials.

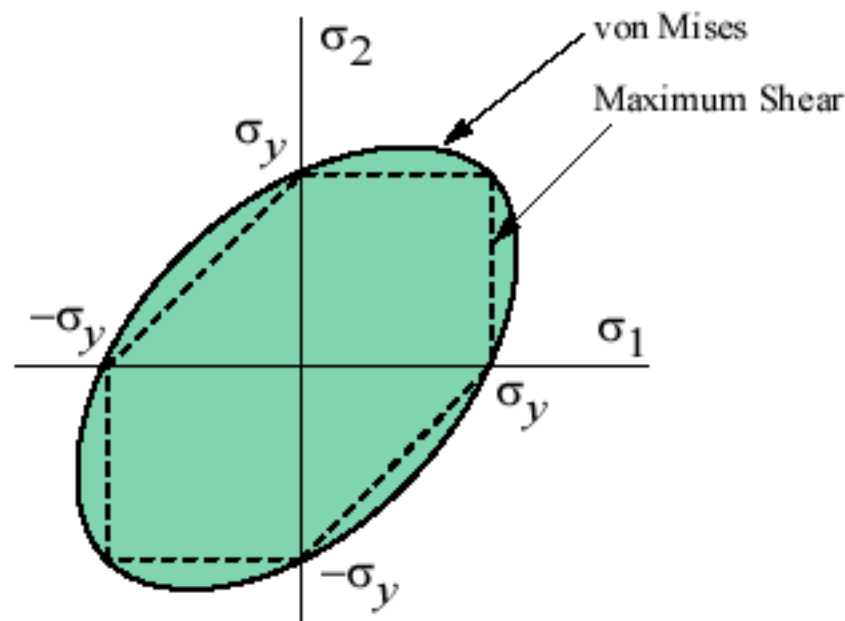
The von Mises criterion states that failure occurs when the energy of distortion reaches the same energy for yield/failure in uniaxial tension. Mathematically, this is expressed as,

$$\frac{1}{2} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \leq \sigma_y^2$$

In the cases of plane stress, $\sigma_3 = 0$. The von Mises criterion reduces to,

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 \leq \sigma_y^2$$

This equation represents a principal stress ellipse as illustrated in the following figure,



Also shown on the figure is the [maximum shear stress criterion](#) (dashed line). This theory is more conservative than the von Mises criterion since it lies inside the von Mises ellipse.

In addition to bounding the principal stresses to prevent ductile failure, the von Mises criterion also gives a reasonable estimation of fatigue failure, especially in cases of repeated tensile and tensile-shear loading.

Failure Criteria for Brittle Materials

Maximum Normal Stress Criterion

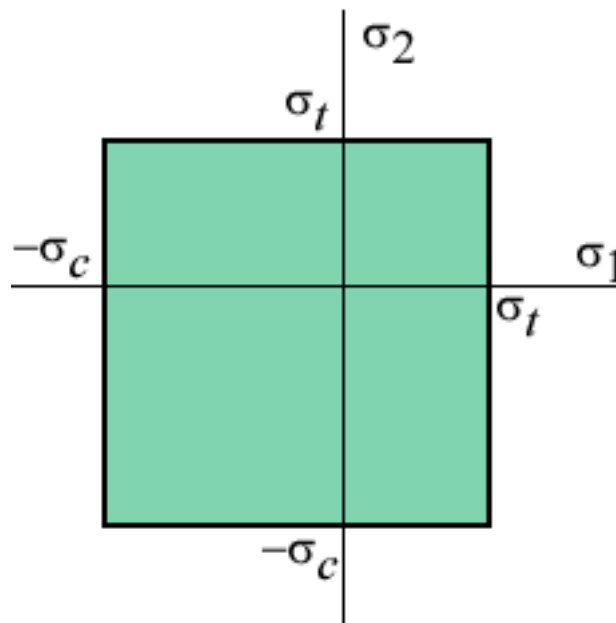
The maximum stress criterion, also known as the normal stress, Coulomb, or Rankine criterion, is often used to predict the failure of brittle materials.

The maximum stress criterion states that failure occurs when the maximum (normal) [principal stress](#) reaches either the *uniaxial* tension strength σ_t , or the *uniaxial* compression strength σ_c ,

$$-\sigma_c < \{\sigma_1, \sigma_2\} < \sigma_t$$

where σ_1 and σ_2 are the principal stresses for 2D stress.

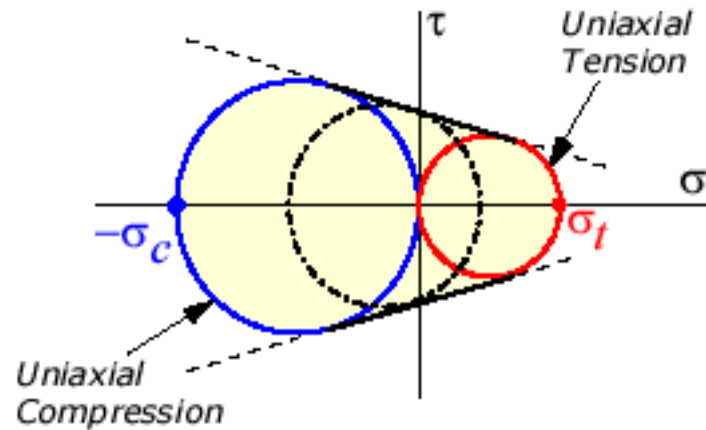
Graphically, the maximum stress criterion requires that the two principal stresses lie within the green zone depicted below,



Mohr's Theory

The Mohr Theory of Failure, also known as the Coulomb-Mohr criterion or internal-friction theory, is based on the famous [Mohr's Circle](#). Mohr's theory is often used in predicting the failure of brittle materials, and is applied to cases of 2D stress.

Mohr's theory suggests that failure occurs when Mohr's Circle at a point in the body exceeds the envelope created by the two Mohr's circles for uniaxial tensile strength and uniaxial compression strength. This envelope is shown in the figure below,



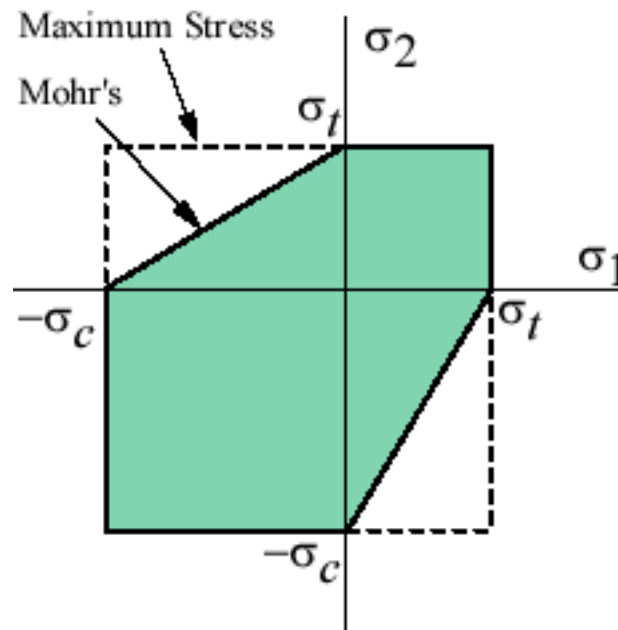
The left circle is for uniaxial compression at the limiting compression stress σ_c of the material. Likewise, the right circle is for uniaxial tension at the limiting tension stress σ_t .

The middle Mohr's Circle on the figure (dash-dot-dash line) represents the maximum allowable stress for an intermediate stress state.

All intermediate stress states fall into one of the four categories in the following table. Each case defines the maximum allowable values for the two principal stresses to avoid failure.

Case	Principal Stresses		Criterion requirements
1	Both in tension	$\sigma_1 > 0, \sigma_2 > 0$	$\sigma_1 < \sigma_t, \sigma_2 < \sigma_t$
2	Both in compression	$\sigma_1 < 0, \sigma_2 < 0$	$\sigma_1 > -\sigma_c, \sigma_2 > -\sigma_c$
3	σ_1 in tension, σ_2 in compression	$\sigma_1 > 0, \sigma_2 < 0$	$\frac{\sigma_1}{\sigma_t} + \frac{\sigma_2}{-\sigma_c} < 1$
4	σ_1 in compression, σ_2 in tension	$\sigma_1 < 0, \sigma_2 > 0$	$\frac{\sigma_1}{-\sigma_c} + \frac{\sigma_2}{\sigma_t} < 1$

Graphically, Mohr's theory requires that the two principal stresses lie within the green zone depicted below,



Also shown on the figure is the [maximum stress criterion](#) (dashed line). This theory is less conservative than Mohr's theory since it lies outside Mohr's boundary.

Techniques for Failure Prevention and Diagnosis

There exist a set of basic techniques for preventing failure in the design stage, and for diagnosing failure in manufacturing and later stages.

In the Design Stage

It is quite commonplace today for design engineers to verify design stresses with finite element (FEA) packages. This is fine and good when FEA is applied appropriately. However, the popularity of finite element analysis can condition engineers to look just for red spots in simulation output, without really understanding the essence or *funda* at play.

By following basic rules of thumb, such danger points can often be anticipated and avoided without total reliance on computer simulation.

Loading Points	Maximum stresses are often located at loading points, supports, joints, or maximum deflection points.
Stress Concentrations	<p>Stress concentrations are usually located near corners, holes, crack tips, boundaries, between layers, and where cross-section areas change rapidly.</p> <p>Sound design avoids rapid changes in material or geometrical properties. For example, when a large hole is removed from a structure, a reinforcement composed of generally no less than the material removed should be added around the opening.</p>
Safety Factors	The addition of safety factors to designs allow engineers to reduce sensitivity to manufacturing defects and to compensate for stress prediction limitations.

In Post-Manufacturing Stages

Despite the best efforts of design and manufacturing engineers, unanticipated failure may occur in parts after design and manufacturing. In order for projects to succeed, these failures must be diagnosed and resolved quickly and effectively. Often, the failure is caused by a singular factor, rather than an involved collection of factors.

Such failures may be caught early in initial quality assurance testing, or later after the part is delivered to the customer.

Induced Stress Concentrations	<p>Stress concentrations may be induced by inadequate manufacturing processes.</p> <p>For example, initial surface imperfections can result from sloppy machining processes. Manufacturing defects such as size mismatches and improper fastener application can lead to residual stresses and even cracks, both strong stress concentrations.</p>
Damage and Exposure	<p>Damages during service life can lead a part to failure. Damages such as cracks, debonding, and delamination can result from unanticipated resonant vibrations and impacts that exceed the design loads.</p> <p>Reduction in strength can result from exposure to UV lights and chemical corrosion.</p>
Fatigue and Creep	<p>Fatigue or creep can lead a part to failure. For example, unanticipated fatigue can result from repeated mechanical or thermal loading.</p>


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27 november 2001	17
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29 november 2001	16
30 november 2001	6
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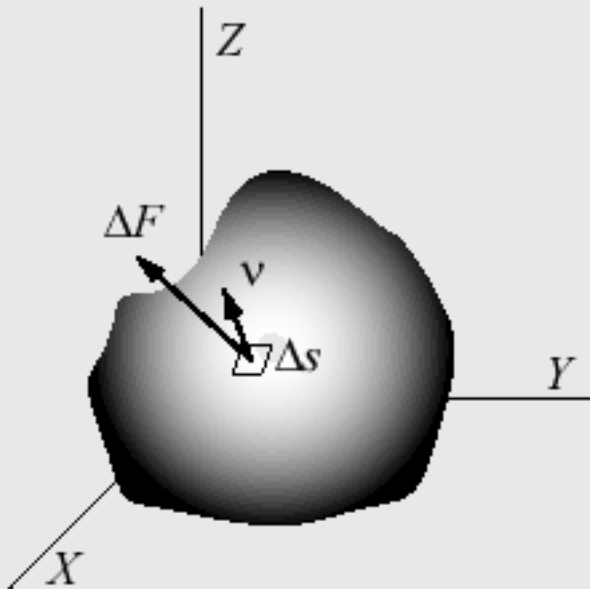
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Solid Mechanics: Stress Introduction

The Definition of Stress

The concept of **stress** originated from the study of strength and failure of solids. The stress field is the distribution of internal "tractions" that balance a given set of external tractions and body forces.



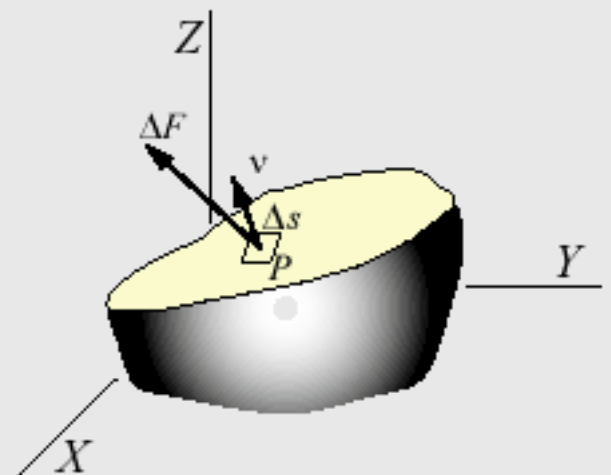
First, we look at the external traction \mathbf{T} that represents the force per unit area acting at a given location on the body's surface. Traction \mathbf{T} is a *bound vector*, which means \mathbf{T} cannot slide along its line of action or translate to another location and keep the same meaning.

In other words, a traction vector cannot be fully described unless both the force and the surface where the force acts on has been specified. Given both ΔF and Δs , the traction \mathbf{T} can be defined as

$$\mathbf{T} = \lim_{\Delta s \rightarrow 0} \frac{\Delta \mathbf{F}}{\Delta s} = \frac{d\mathbf{F}}{ds}$$

The internal traction within a solid, or stress, can be defined in a similar manner. Suppose an arbitrary slice is made across the solid shown in the above figure, leading to the free body diagram shown at right. Surface tractions would appear on the exposed surface, similar in form to the external tractions applied to the body's exterior surface. The stress at point P can be defined using the same equation as was used for \mathbf{T} .

Stress therefore can be interpreted as internal tractions that act on a defined internal datum plane. One cannot measure the stress without first specifying the datum plane.



The Stress Tensor (or Stress Matrix)

Surface tractions, or stresses acting on an internal datum plane, are typically decomposed into three mutually orthogonal components. One component is normal to the surface and represents *direct stress*. The other two components are tangential to the surface and represent *shear stresses*.

What is the distinction between normal and tangential tractions, or equivalently, direct and shear stresses? **Direct stresses** tend to change the volume of the material (e.g. hydrostatic pressure) and are resisted by the body's bulk modulus (which depends on the Young's modulus and Poisson ratio). **Shear stresses** tend to deform the material without changing its volume, and are resisted by the body's shear modulus.

Defining a set of internal datum planes aligned with a Cartesian coordinate system allows the stress state at an internal point *P* to be described relative to *x*, *y*, and *z* coordinate directions.

For example, the stress state at point *P* can be represented by an *infinitesimal* cube with three stress components on each of its six sides (one direct and two shear components).

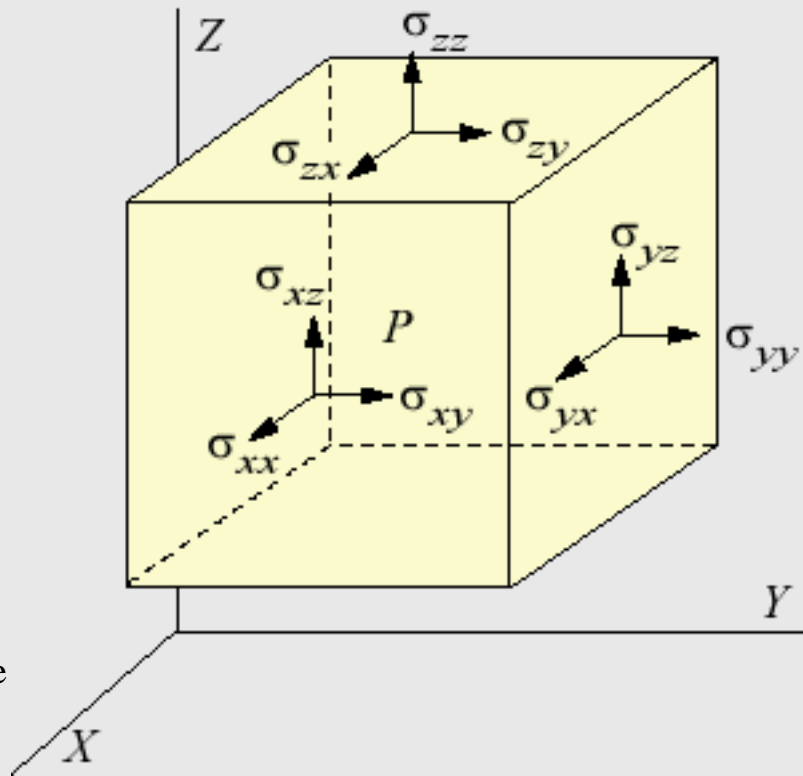
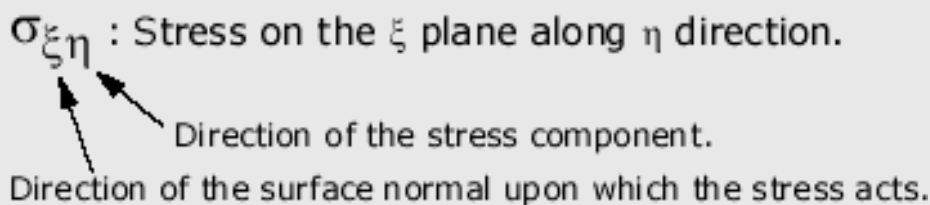
Since each point in the body is under static equilibrium (no net force in the absence of any body forces), only nine stress components from three planes are needed to describe the stress state at a point *P*.

These nine components can be organized into the matrix:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

where shear stresses across the diagonal are identical (i.e. $\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{zy}$, and $\sigma_{zx} = \sigma_{xz}$) as a result of static equilibrium (no net moment). This grouping of the nine stress components is known as the **stress tensor** (or stress matrix).

The subscript notation used for the nine stress components have the following meaning:



Note: The stress state is a second order tensor since it is a quantity associated with two directions. As a result, stress components have 2 subscripts.

A surface traction is a first order tensor (i.e. vector) since it a quantity associated with only one direction. Vector components therefore require only 1 subscript.

Mass would be an example of a zero-order tensor (i.e. scalars), which have no relationships with directions (and no subscripts).

Equations of Equilibrium

Consider the static equilibrium of a solid subjected to the body force vector field \mathbf{b} . Applying Newton's first law of motion results in the following set of differential equations which govern the stress distribution within the solid,

$$\begin{cases} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + b_x = 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + b_y = 0 \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0 \end{cases}$$

In the case of two dimensional stress, the above equations reduce to,

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + b_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0 \end{cases}$$

Solid Mechanics: Stress

Plane Stress and Coordinate Transformations

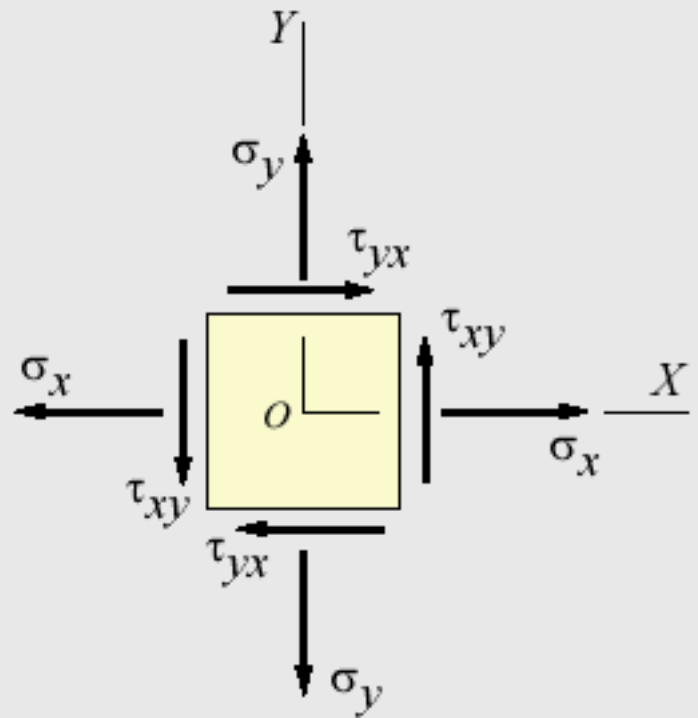
Plane State of Stress

A class of common engineering problems involving stresses in a thin plate or on the free surface of a structural element, such as the surfaces of thin-walled pressure vessels under external or internal pressure, the free surfaces of shafts in torsion and beams under transverse load, have one [principal stress](#) that is much smaller than the other two. By assuming that this small principal stress is zero, the three-dimensional stress state can be reduced to two dimensions. Since the remaining two principal stresses lie in a plane, these simplified 2D problems are called **plane stress** problems.

Assume that the negligible principal stress is oriented in the z -direction. To reduce the [3D stress matrix](#) to the 2D plane stress matrix, remove all components with z subscripts to get,

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

where $\tau_{xy} = \tau_{yx}$ for static equilibrium. The sign convention for positive stress components in plane stress is illustrated in the above figure on the 2D element.



Coordinate Transformations

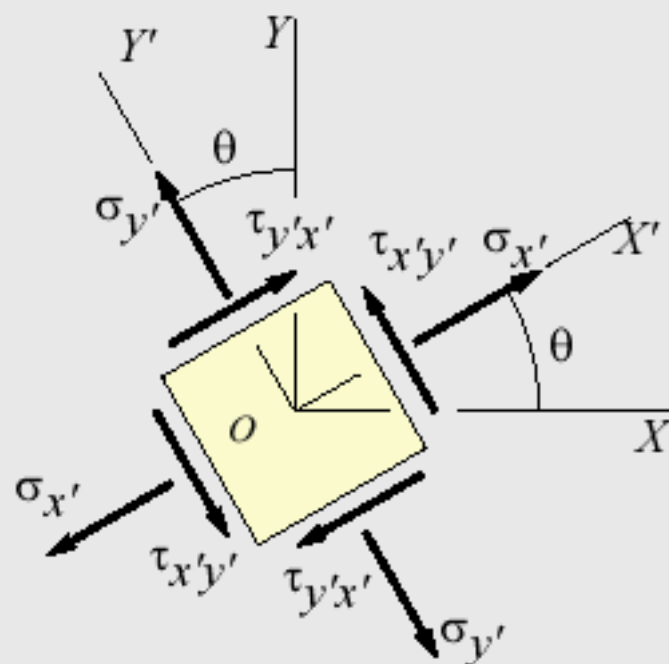
The coordinate directions chosen to analyze a structure are usually based on the shape of the structure. As a result, the direct and shear stress components are associated with these directions. For example, to analyze a bar one almost always directs one of the coordinate directions along the bar's axis.

Nonetheless, stresses in directions that do not line up with the original coordinate set are also important. For example, the failure plane of a brittle shaft under torsion is often at a 45° angle with respect to the shaft's axis. Stress transformation formulas are required to analyze these stresses.

The transformation of stresses with respect to the $\{x,y,z\}$ coordinates to the stresses with respect to $\{x',y',z'\}$ is performed via the equations,

$$\left\{ \begin{array}{l} \sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\ \sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \\ \qquad = \sigma_x + \sigma_y - \sigma_{x'} \\ \tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \end{array} \right.$$

where θ is the rotation angle between the two coordinate sets (positive in the counterclockwise direction). This angle along with the stresses for the $\{x',y',z'\}$ coordinates are shown in the figure below,



Solid Mechanics: Stress

Principal Stress for the Case of Plane Stress

Principal Directions, Principal Stress

The normal stresses ($\sigma_{x'}$ and $\sigma_{y'}$) and the shear stress ($\tau_{x'y'}$) vary smoothly with respect to the rotation angle θ , in accordance with the [coordinate transformation](#) equations. There exist a couple of particular angles where the stresses take on special values.

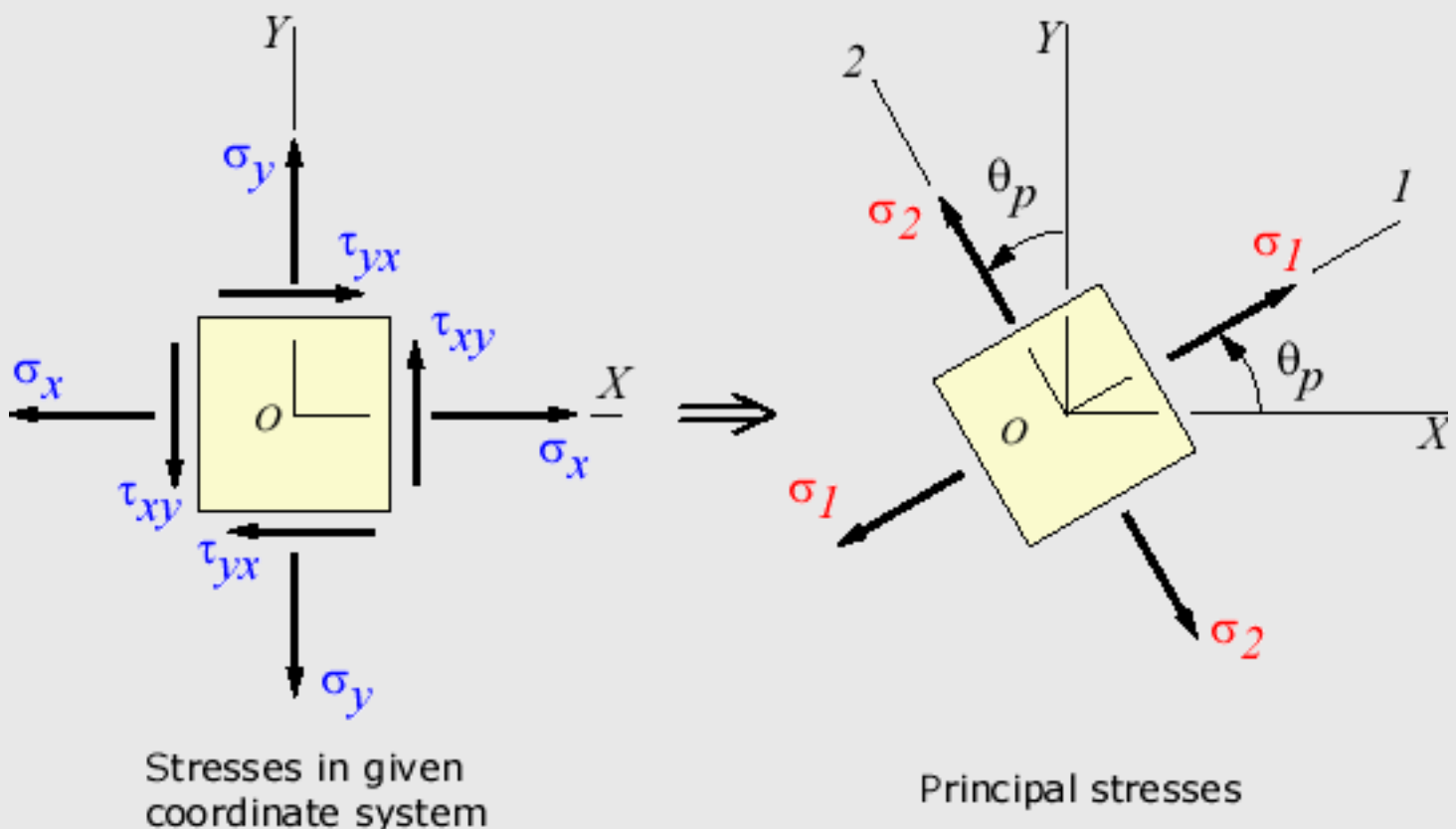
First, there exists an angle θ_p where the shear stress $\tau_{x'y'}$ becomes zero. That angle is found by setting $\tau_{x'y'}$ to zero in the above shear transformation equation and solving for θ (set equal to θ_p). The result is,

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

The angle θ_p defines the *principal directions* where the only stresses are normal stresses. These stresses are called *principal stresses* and are found from the original stresses (expressed in the x,y,z directions) via,

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

The transformation to the principal directions can be illustrated as:



Maximum Shear Stress Direction

Another important angle, θ_s , is where the maximum shear stress occurs. This is found by finding the maximum of the shear stress transformation equation, and solving for θ . The result is,

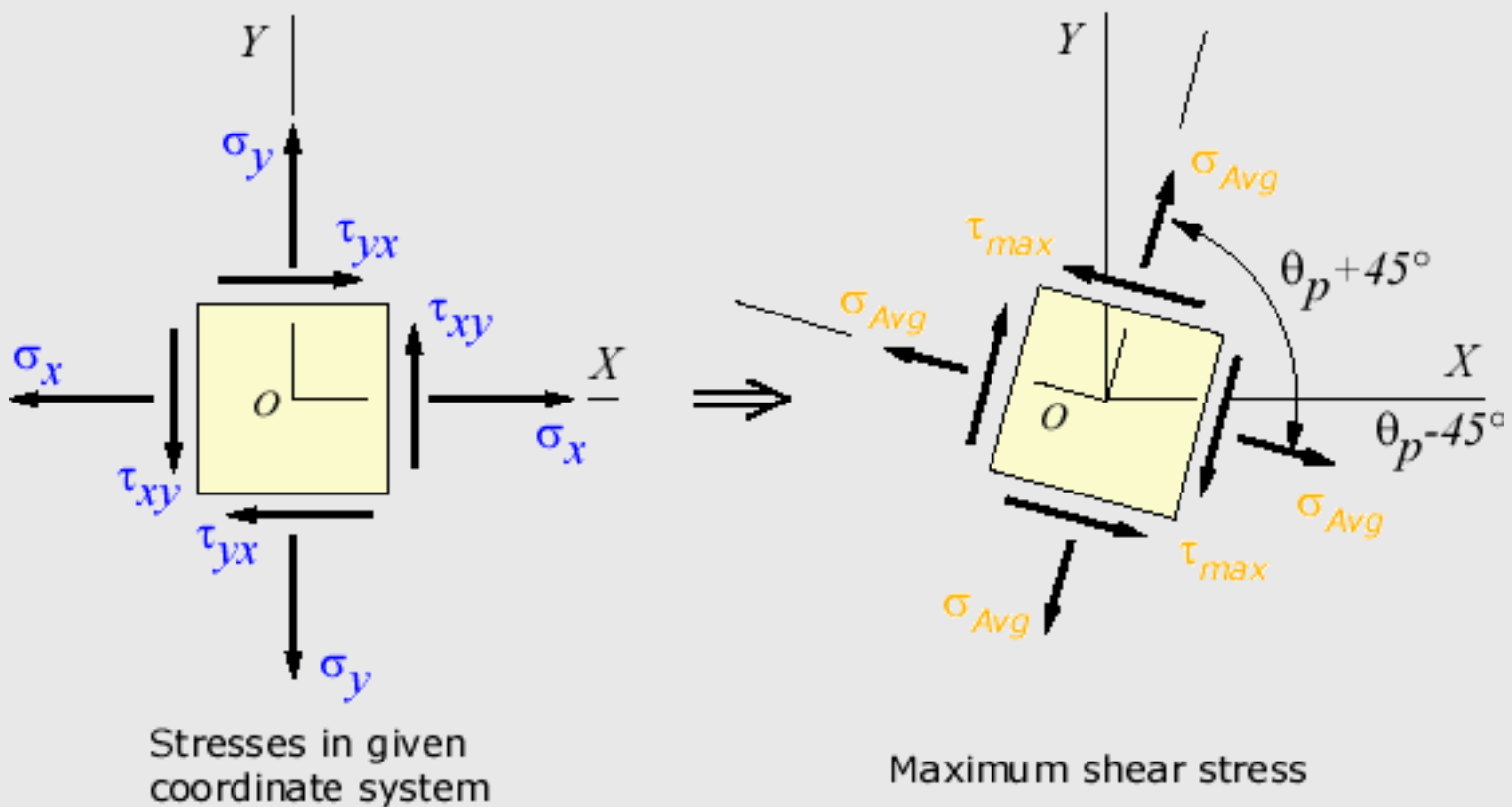
$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\Rightarrow \theta_s = \theta_p \pm 45^\circ$$

The maximum shear stress is equal to one-half the difference between the two principal stresses,

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_1 - \sigma_2}{2}$$

The transformation to the maximum shear stress direction can be illustrated as:

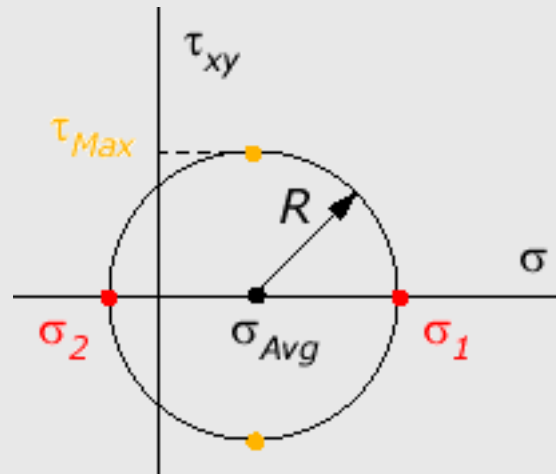


Solid Mechanics: Stress

Mohr's Circle for Plane Stress

Mohr's Circle

Introduced by Otto Mohr in 1882, Mohr's Circle illustrates principal stresses and stress transformations via a graphical format,



The two principal stresses are shown in **red**, and the maximum shear stress is shown in **orange**. Recall that the normal stresses equal the principal stresses when the stress element is aligned with the principal directions, and the shear stress equals the maximum shear stress when the stress element is rotated 45° away from the principal directions.

As the stress element is rotated away from the [principal](#) (or maximum shear) directions, the normal and shear stress components will always lie on Mohr's Circle.

Mohr's Circle was the leading tool used to visualize relationships between normal and shear stresses, and to estimate the maximum stresses, before hand-held calculators became popular. Even today, Mohr's Circle is still widely used by engineers all over the world.

Derivation of Mohr's Circle

To establish Mohr's Circle, we first recall the stress [transformation formulas](#) for plane stress at a given location,

$$\begin{cases} \sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\ \tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \end{cases}$$

Using a basic trigonometric relation ($\cos^2 2\theta + \sin^2 2\theta = 1$) to combine the two above equations we have,

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

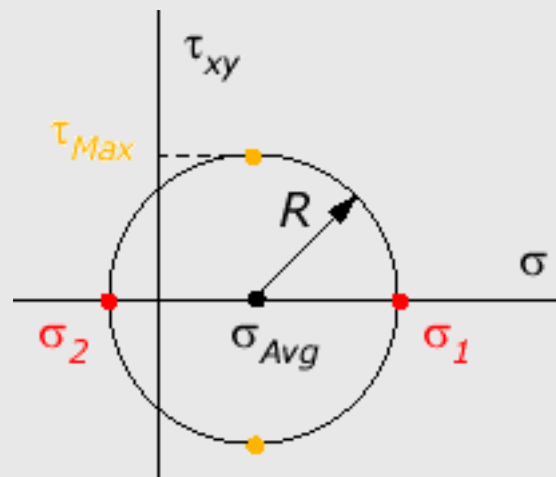
This is the equation of a circle, plotted on a graph where the abscissa is the normal stress and the ordinate is the shear stress. This is easier to see if we interpret σ_x and σ_y as being the two [principal stresses](#), and τ_{xy} as being the maximum shear stress. Then we can define the average stress, σ_{avg} , and a "radius" R (which is just equal to the maximum shear stress),

$$\sigma_{Avg} = \frac{\sigma_x + \sigma_y}{2} \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

The circle equation above now takes on a more familiar form,

$$\left(\sigma_{x'} - \sigma_{Avg}\right)^2 + \tau_{x'y'}^2 = R^2$$

The circle is centered at the average stress value, and has a radius R equal to the maximum shear stress, as shown in the figure below,



Related Topics

The procedure of drawing a Mohr's Circle from a given stress state is discussed in the [Mohr's Circle usage](#) page.

The Mohr's Circle for [plane strain](#) can also be obtained from similar procedures.

Solid Mechanics: Stress

Mohr's Circle Usage in Plane Stress

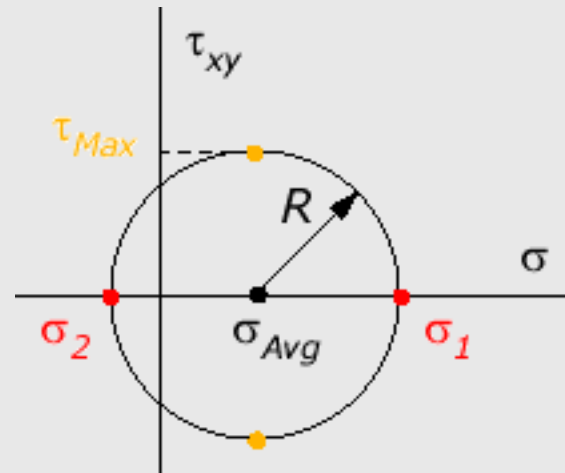
Principal Stresses from Mohr's Circle

A chief benefit of Mohr's circle is that the [principal stresses](#) σ_1 and σ_2 and the maximum shear stress τ_{max} are obtained immediately after drawing the circle,

$$\begin{cases} \sigma_{1,2} = \sigma_{Avg} \pm R \\ \tau_{Max} = R \end{cases}$$

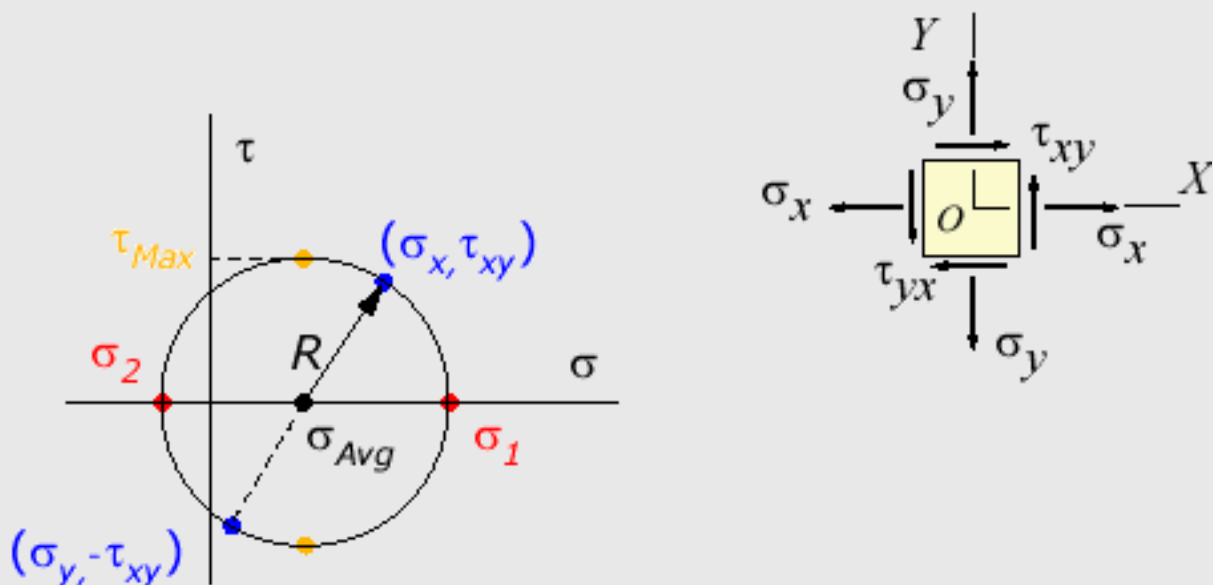
where,

$$\sigma_{Avg} = \frac{\sigma_x + \sigma_y}{2} \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



Principal Directions from Mohr's Circle

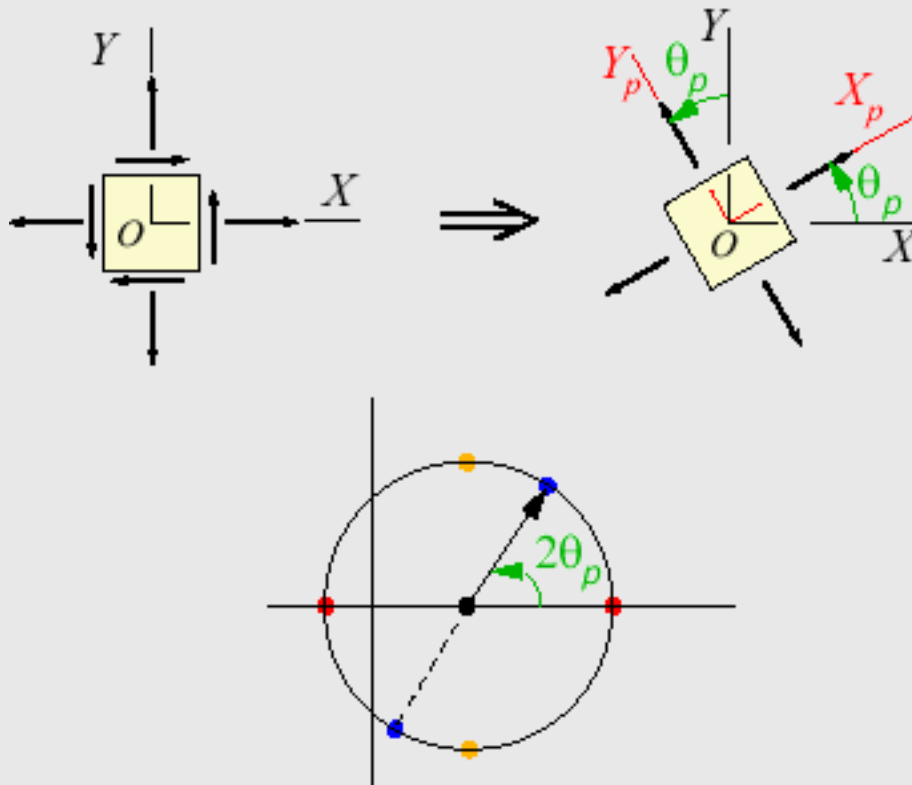
Mohr's Circle can be used to find the directions of the principal axes. To show this, first suppose that the normal and shear stresses, σ_x , σ_y , and τ_{xy} , are obtained at a given point O in the body. They are expressed relative to the coordinates XY , as shown in the stress element at right below.



The Mohr's Circle for this general stress state is shown at left above. Note that it's centered at σ_{avg} and has a radius R , and that the two points $\{\sigma_x, \tau_{xy}\}$ and $\{\sigma_y, -\tau_{xy}\}$ lie on opposite sides of the circle. The line connecting σ_x and σ_y will be defined as L_{xy} .

The **angle** between the current axes (X and Y) and the **principal axes** is defined as θ_p , and is equal to

one half the angle between the line L_{xy} and the σ -axis as shown in the schematic below,



A set of six Mohr's Circles representing most stress state possibilities are presented on the [examples](#) page.

Rotation Angle on Mohr's Circle

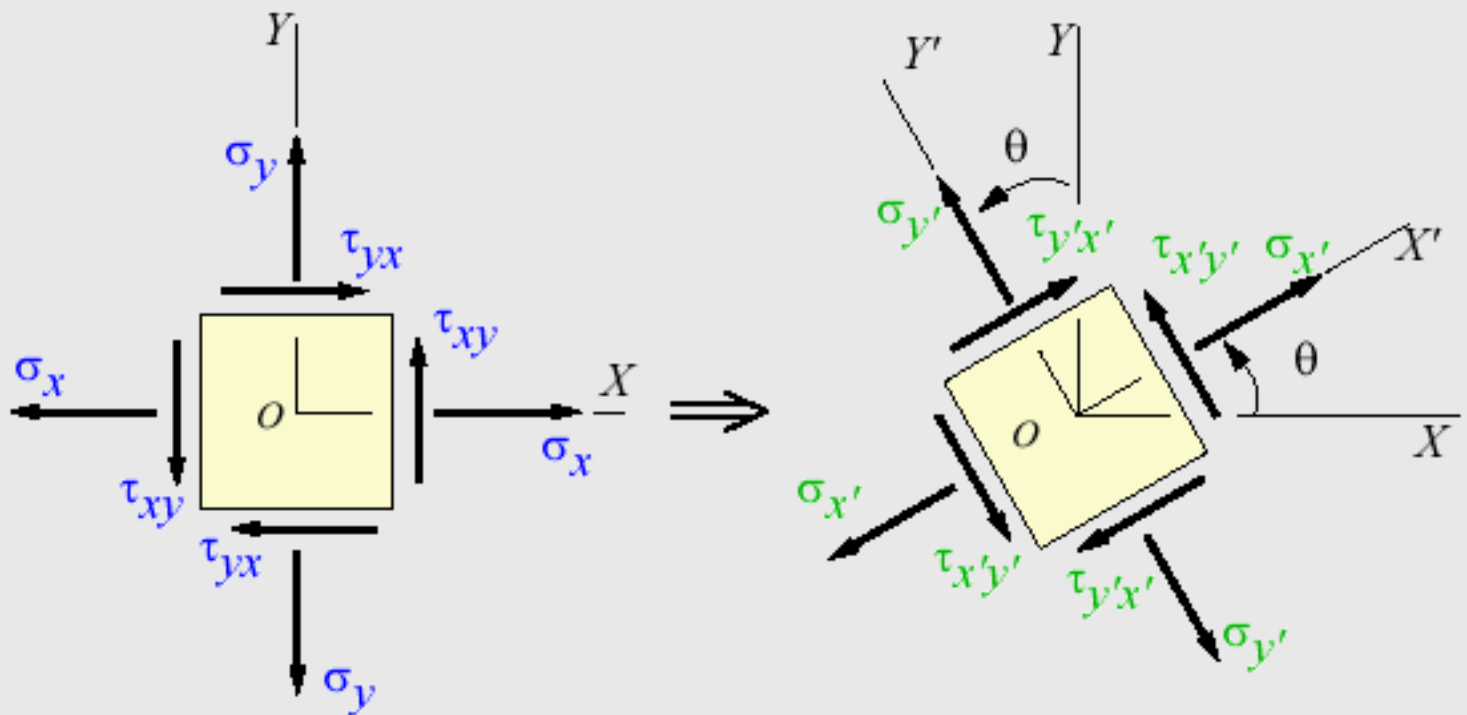
Note that the coordinate rotation angle θ_p is defined positive when starting at the XY coordinates and proceeding to the X_pY_p coordinates. In contrast, on the Mohr's Circle θ_p is defined positive starting on the principal stress line (i.e. the σ -axis) and proceeding to the XY stress line (i.e. line L_{xy}). The angle θ_p has the opposite sense between the two figures, because on one it starts on the XY coordinates, and on the other it starts on the principal coordinates.

Some books avoid this dichotomy between θ_p on Mohr's Circle and θ_p on the stress element by locating $(\sigma_x, -\tau_{xy})$ instead of (σ_x, τ_{xy}) on Mohr's Circle. This will switch the polarity of θ_p on the circle. Whatever method you choose, the bottom line is that an *opposite* sign is needed either in the interpretation or in the plotting to make Mohr's Circle physically meaningful.

Stress Transform by Mohr's Circle

Mohr's Circle can be used to transform stresses from one coordinate set to another, similar that that described on the [plane stress](#) page.

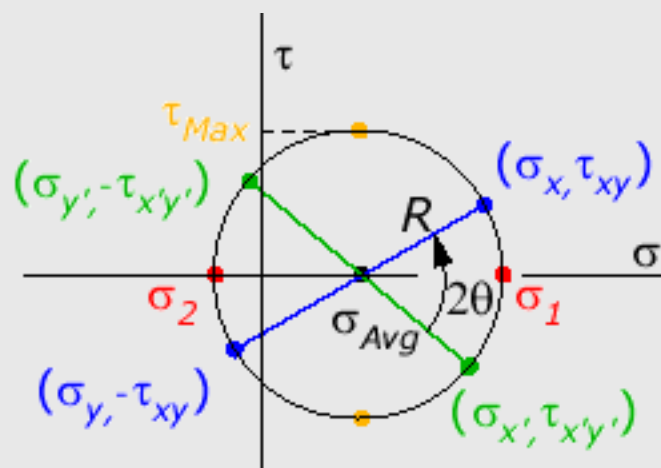
Suppose that the normal and shear stresses, σ_x , σ_y , and τ_{xy} , are obtained at a point O in the body, expressed with respect to the coordinates XY . We wish to find the stresses expressed in the new coordinate set $X'Y'$, rotated an angle θ from XY , as shown below:



Stresses at given coordinate system Stresses transformed to another coordinate

To do this we proceed as follows:

- Draw Mohr's circle for the **given stress state** (σ_x , σ_y , and τ_{xy} ; shown below).
- Draw the line L_{xy} across the circle from (σ_x, τ_{xy}) to $(\sigma_y, -\tau_{xy})$.
- Rotate the line L_{xy} by $2*\theta$ (twice as much as the angle between XY and $X'Y'$) and in the *opposite* direction of θ .
- The **stresses in the new coordinates** ($\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$) are then read off the circle.

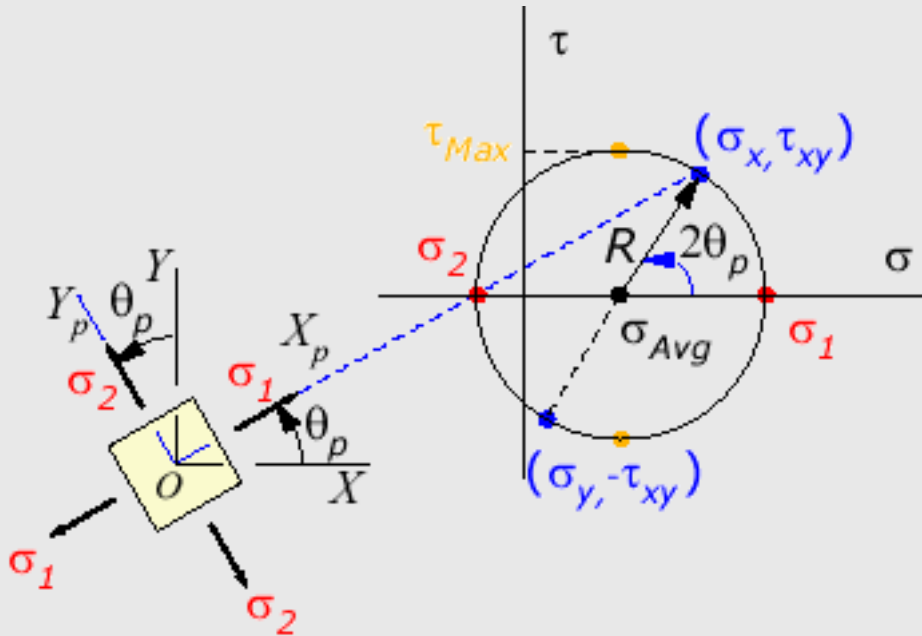


Solid Mechanics: Stress

Examples of Mohr's Circles in Plane Stress

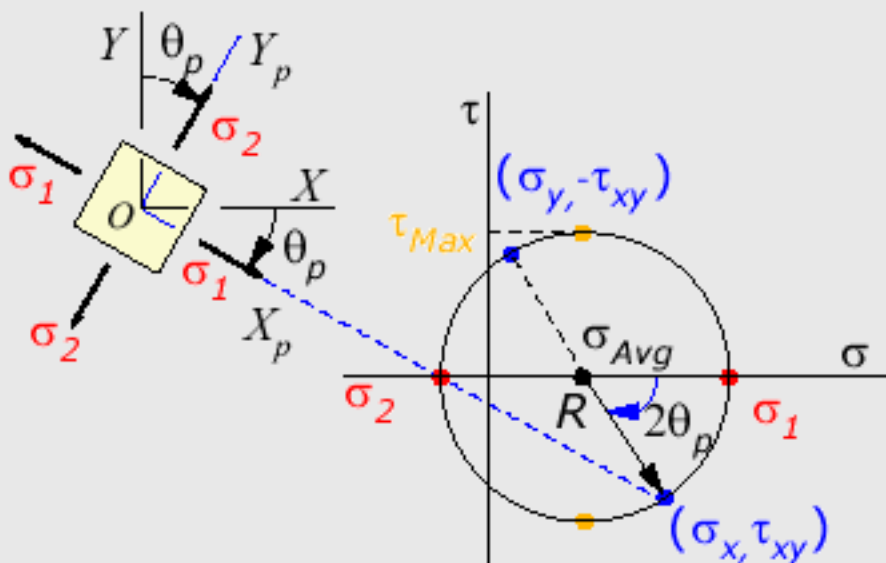
Case 1: $\tau_{xy} > 0$ and $\sigma_x > \sigma_y$

The principal axes are counterclockwise to the current axes (because $\tau_{xy} > 0$) and no more than 45° away (because $\sigma_x > \sigma_y$).



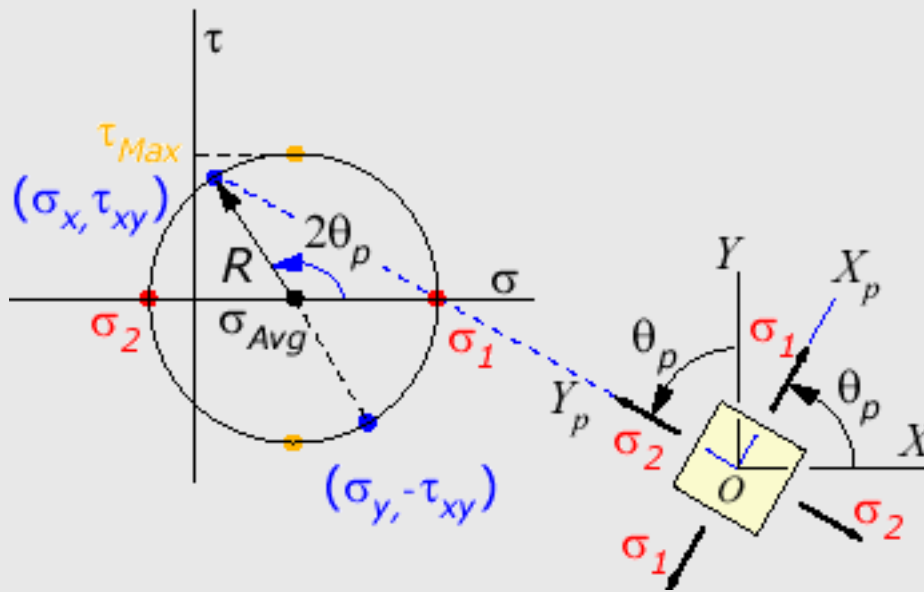
Case 2: $\tau_{xy} < 0$ and $\sigma_x > \sigma_y$

The principal axes are clockwise to the current axes (because $\tau_{xy} < 0$) and no more than 45° away (because $\sigma_x > \sigma_y$).

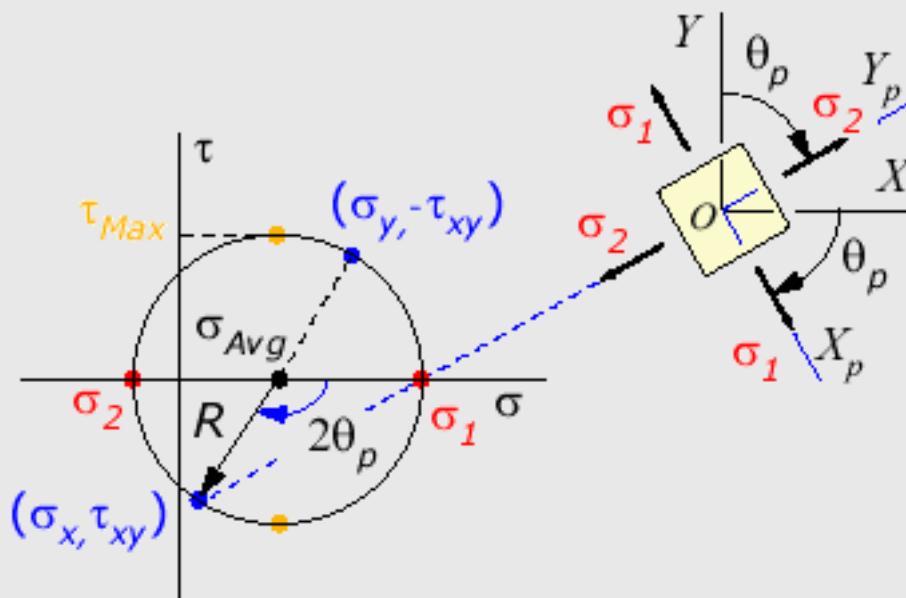


Case 3: $\tau_{xy} > 0$ and $\sigma_x < \sigma_y$

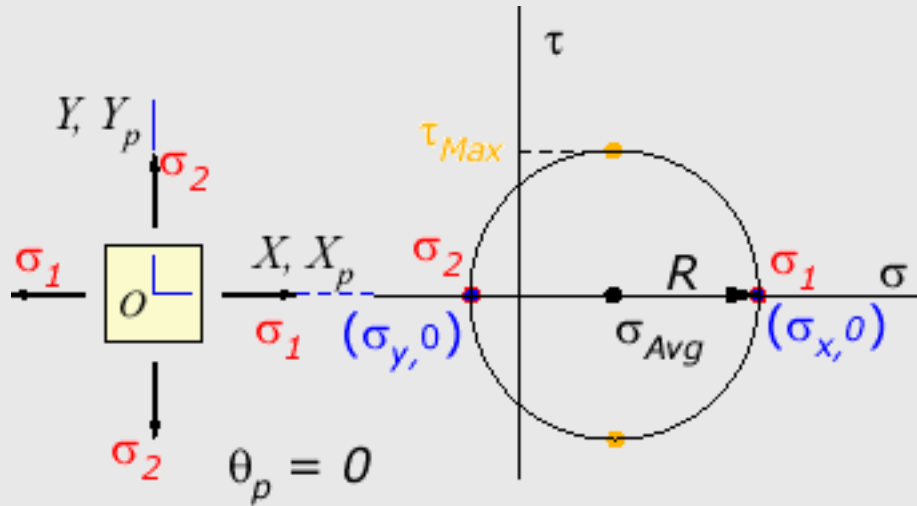
The principal axes are counterclockwise to the current axes (because $\tau_{xy} > 0$) and between 45° and 90° away (because $\sigma_x < \sigma_y$).

**Case 4: $\tau_{xy} < 0$ and $\sigma_x < \sigma_y$**

The principal axes are clockwise to the current axes (because $\tau_{xy} < 0$) and between 45° and 90° away (because $\sigma_x < \sigma_y$).

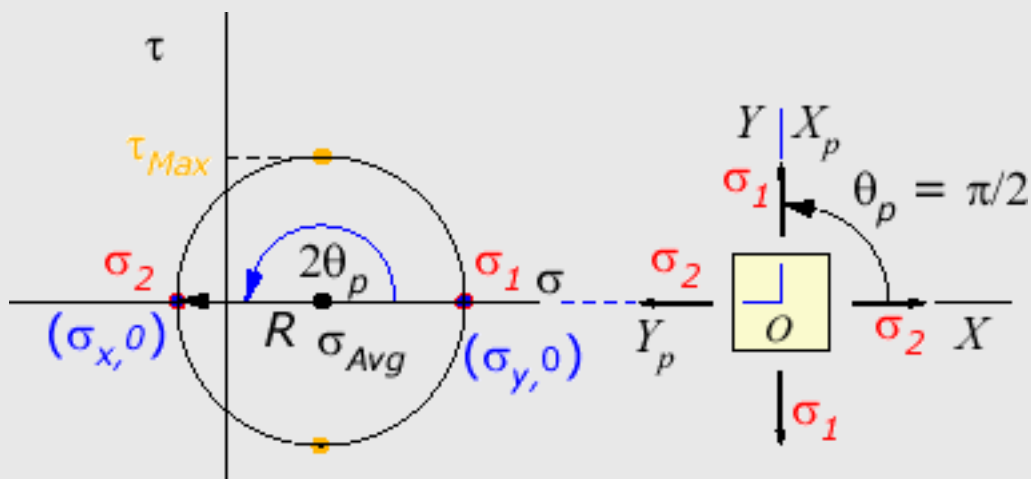
**Case 5: $\tau_{xy} = 0$ and $\sigma_x > \sigma_y$**

The principal axes are aligned with the current axes (because $\sigma_x > \sigma_y$ and $\tau_{xy} = 0$).



Case 6: $\tau_{xy} = 0$ and $\sigma_x < \sigma_y$

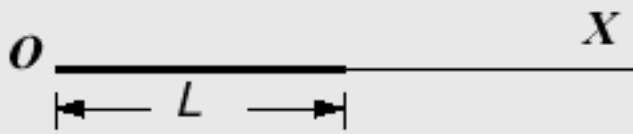
The principal axes are exactly 90° from the current axes (because $\sigma_x < \sigma_y$ and $\tau_{xy} = 0$).



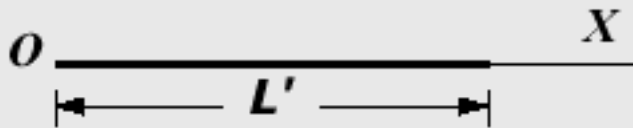
Solid Mechanics: Strain

Introduction

Global 1D Strain



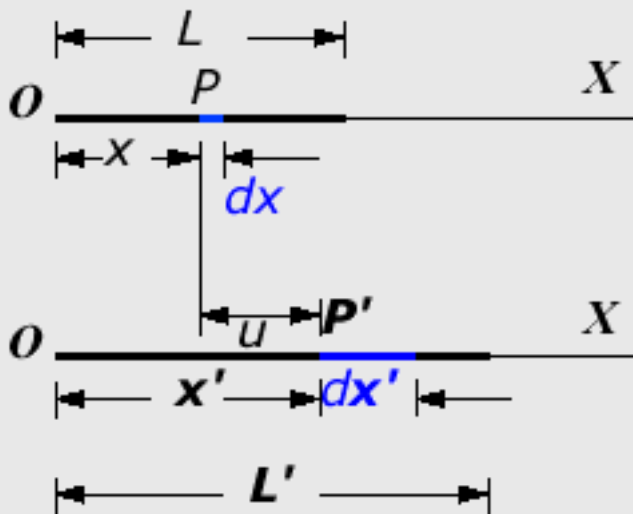
Consider a rod with initial length L which is stretched to a length L' . The strain measure ε , a dimensionless ratio, is defined as the ratio of elongation with respect to the original length,



$$\varepsilon = \frac{L' - L}{L}$$

Infinitesimal 1D Strain

The above strain measure is defined in a global sense. The strain at each point may vary dramatically if the bar's elastic modulus or cross-sectional area changes. To track down the strain at each point, further refinement in the definition is needed.



Consider an arbitrary point in the bar P , which has a position vector x , and its infinitesimal neighbor dx . Point P shifts to P' , which has a position vector x' , after the stretch. In the meantime, the small "step" dx is stretched to dx' .

The strain at point p can be defined the same as in the global strain measure,

$$\varepsilon = \frac{dx' - dx}{dx}$$

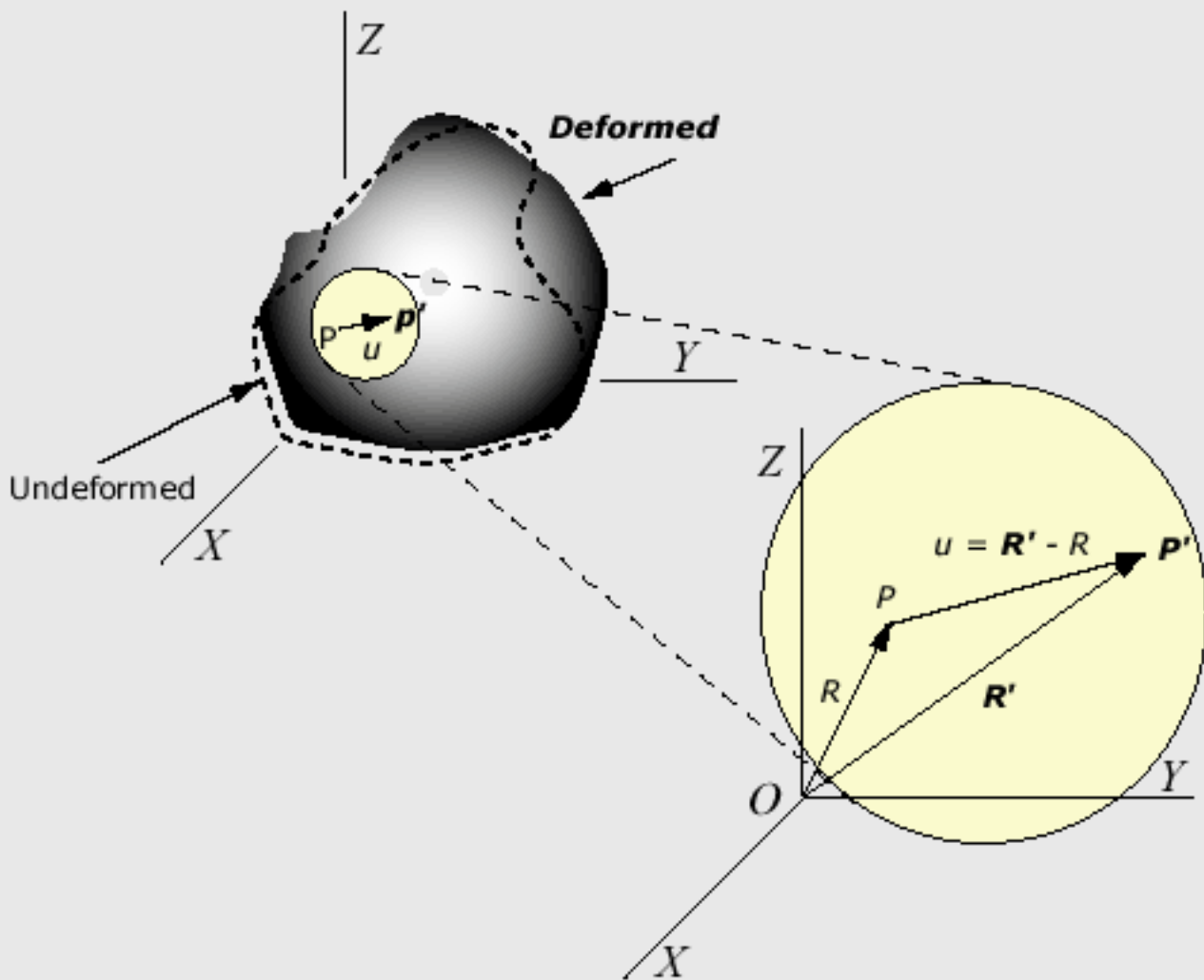
Since the displacement $u = x' - x$, the strain can

hence be rewritten as,

$$\varepsilon = \frac{dx' - dx}{dx} = \frac{du}{dx}$$

General Definition of 3D Strain

As in the one dimensional strain derivation, suppose that point P in a body shifts to point P' after deformation.



The infinitesimal strain-displacement relationships can be summarized as,

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

where \mathbf{u} is the displacement vector, x is coordinate, and the two indices i and j can range over the three coordinates $\{1, 2, 3\}$ in three dimensional space.

Expanding the above equation for each coordinate direction gives,

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} & \varepsilon_{yz} &= \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) = \varepsilon_{zy} \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} & \varepsilon_{zx} &= \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \varepsilon_{xz} \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} & \varepsilon_{xy} &= \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = \varepsilon_{yx} \end{aligned}$$

where u , v , and w are the displacements in the x , y , and z directions respectively (i.e. they are the

components of \mathbf{u}).

3D Strain Matrix

There are a total of 6 strain measures. These 6 measures can be organized into a matrix (similar in form to the [3D stress matrix](#)), shown here,

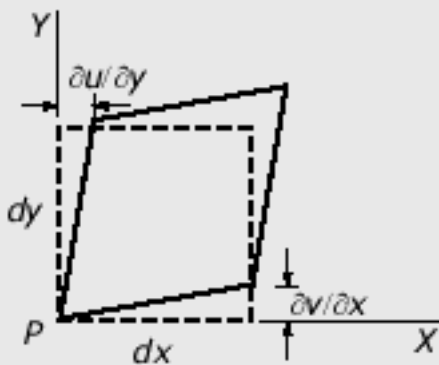
$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

Engineering Shear Strain

Focus on the strain ε_{xy} for a moment. The expression inside the parentheses can be rewritten as,

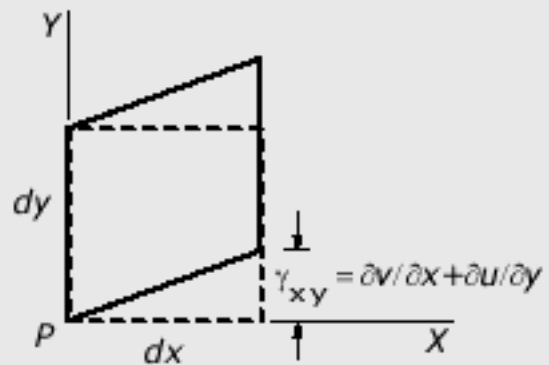
$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$. Called the **engineering shear strain**, γ_{xy} is a total measure of shear strain in the x - y plane. In contrast, the shear strain ε_{xy} is the average of the shear strain on the x face along the y direction, and on the y face along the x direction.



Shear strain tensor is the **average** of two strains, i.e.,

$$\varepsilon_{xy} = (\partial v / \partial x + \partial u / \partial y) / 2 = \varepsilon_{yx}$$



Engineer shear strain is the **total** shear strain, i.e.,

$$\gamma_{xy} = \partial v / \partial x + \partial u / \partial y$$

Engineering shear strain is commonly used in engineering reference books. However, please beware of the difference between shear strain and engineering shear strain, so as to avoid errors in mathematical manipulations.

Compatibility Conditions

In the strain-displacement relationships, there are six strain measures but only three independent displacements. That is, there are 6 unknowns for only 3 independent variables. As a result there exist 3 constraint, or compatibility, equations.

These compatibility conditions for infinitesimal strain referred to rectangular Cartesian coordinates are,

$$\begin{aligned} \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} &= 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} & \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} &= \frac{\partial}{\partial x} \left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right) \\ \frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} &= 2 \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} & \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right) \\ \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} &= 2 \frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} & \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} &= \frac{\partial}{\partial z} \left(\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z} \right) \end{aligned}$$

In two dimensional problems (e.g. [plane strain](#)), all z terms are set to zero. The compatibility equations reduce to,

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$$

Note that some references use engineering shear strain ($\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$) when referencing compatibility equations.

Solid Mechanics: Strain

Plane Strain and Coordinate Transformations

Plane State of Strain

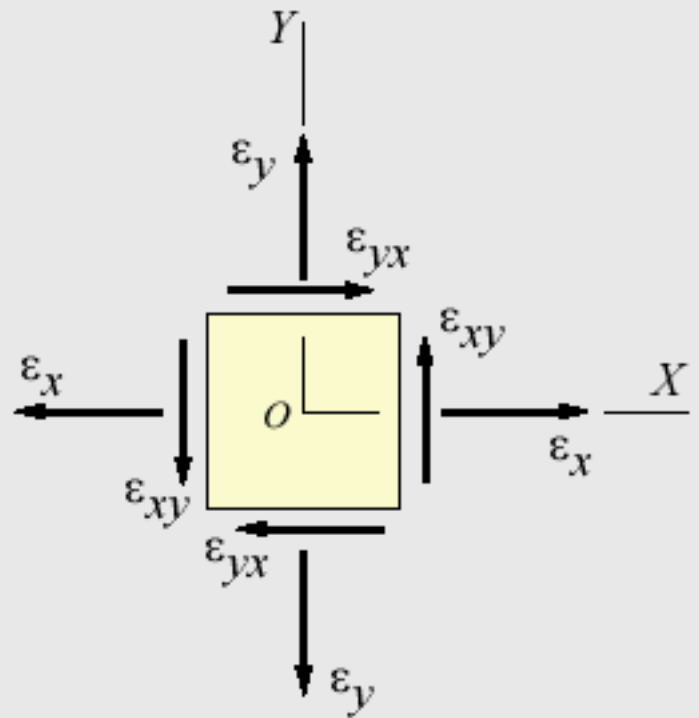
Some common engineering problems such as a dam subjected to water loading, a tunnel under external pressure, a pipe under internal pressure, and a cylindrical roller bearing compressed by force in a diametral plane, have significant strain only in a plane; that is, the strain in one direction is much less than the strain in the two other orthogonal directions. If small enough, the smallest strain can be ignored and the part is said to experience **plane strain**.

Assume that the negligible strain is oriented in the z -direction. To reduce the [3D strain matrix](#) to the 2D plane stress matrix, remove all components with z subscripts to get,

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_y \end{bmatrix}$$

where $\varepsilon_{xy} = \varepsilon_{yx}$ by definition.

The sign convention here is consistent with the sign convention used in [plane stress](#) analysis.

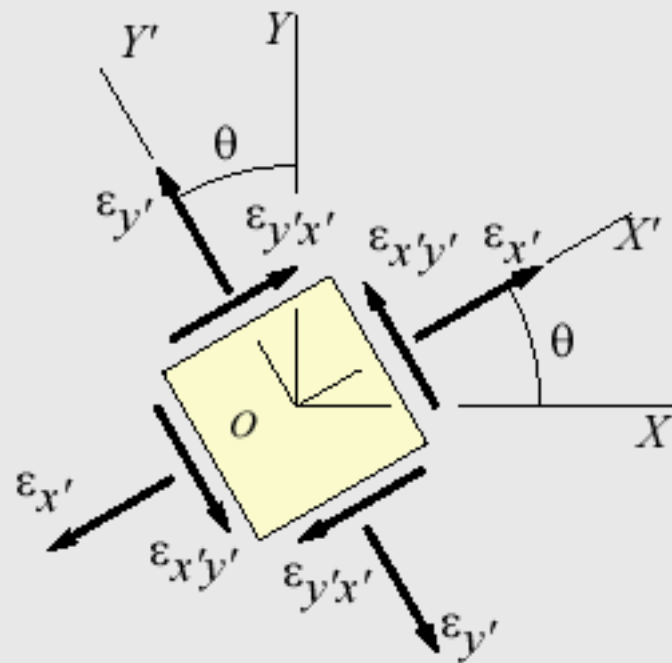


Coordinate Transformation

The transformation of strains with respect to the $\{x,y,z\}$ coordinates to the strains with respect to $\{x',y',z'\}$ is performed via the equations,

$$\begin{cases} \varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \varepsilon_{xy} \sin 2\theta \\ \varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta - \varepsilon_{xy} \sin 2\theta \\ \quad = \varepsilon_x + \varepsilon_y - \varepsilon_{x'} \\ \varepsilon_{x'y'} = -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \varepsilon_{xy} \cos 2\theta \end{cases}$$

The rotation between the two coordinate sets is shown here,



where θ is defined positive in the counterclockwise direction.

Solid Mechanics: Strain

Principal Strain for the Case of Plane Strain

Principal Directions, Principal Strain

The normal strains (ϵ_x' and ϵ_y') and the shear strain ($\epsilon_{x'y'}$) vary smoothly with respect to the rotation angle θ , in accordance with the transformation equations given above. There exist a couple of particular angles where the strains take on special values.

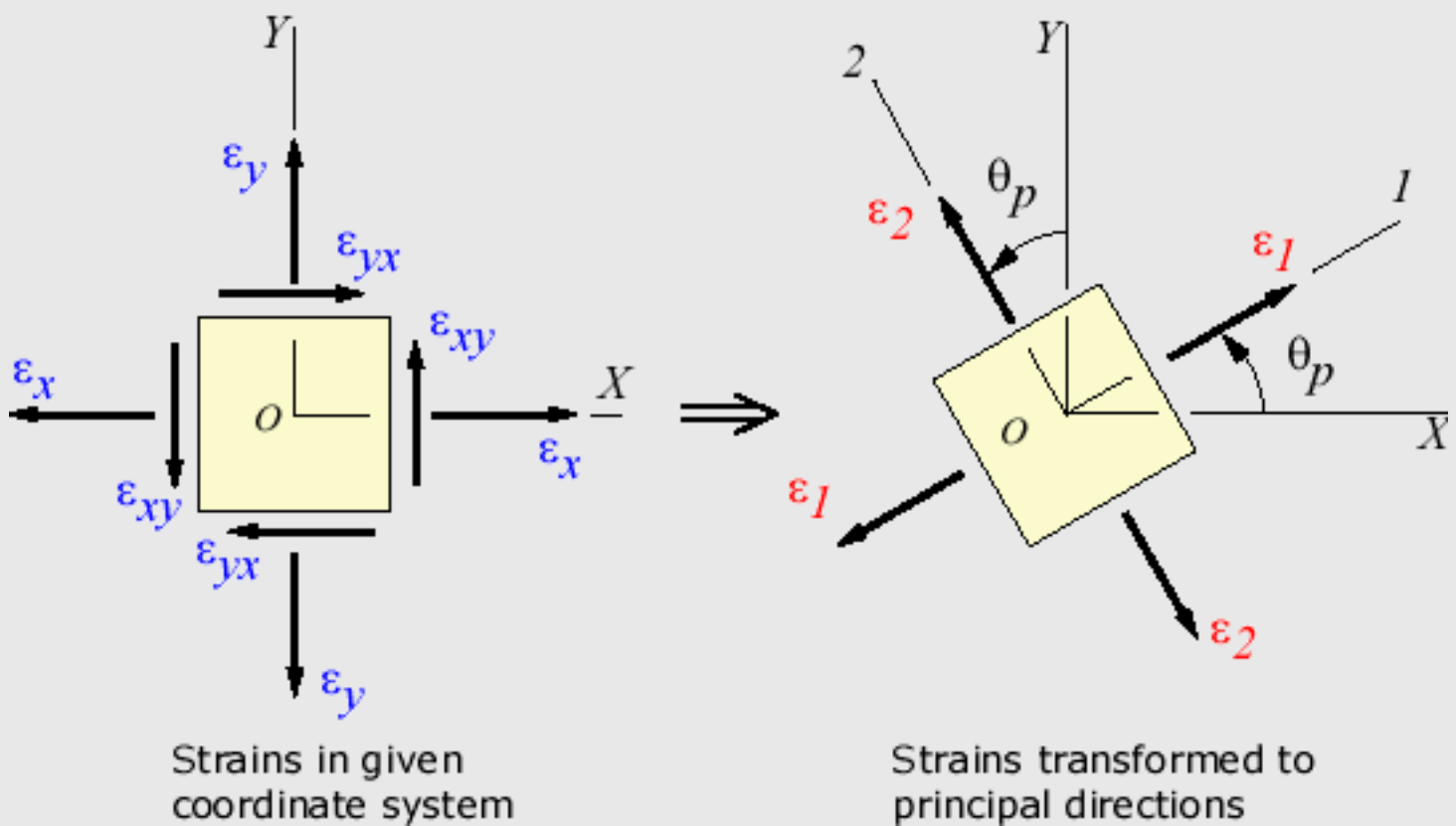
First, there exists an angle θ_p where the shear strain $\epsilon_{x'y'}$ vanishes. That angle is given by,

$$\tan 2\theta_p = \frac{2\epsilon_{xy}}{\epsilon_x - \epsilon_y}$$

This angle defines the *principal directions*. The associated *principal strains* are given by,

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \epsilon_{xy}^2}$$

The transformation to the principal directions with their principal strains can be illustrated as:



Maximum Shear Strain Direction

Another important angle, θ_s , is where the maximum shear strain occurs and is given by,

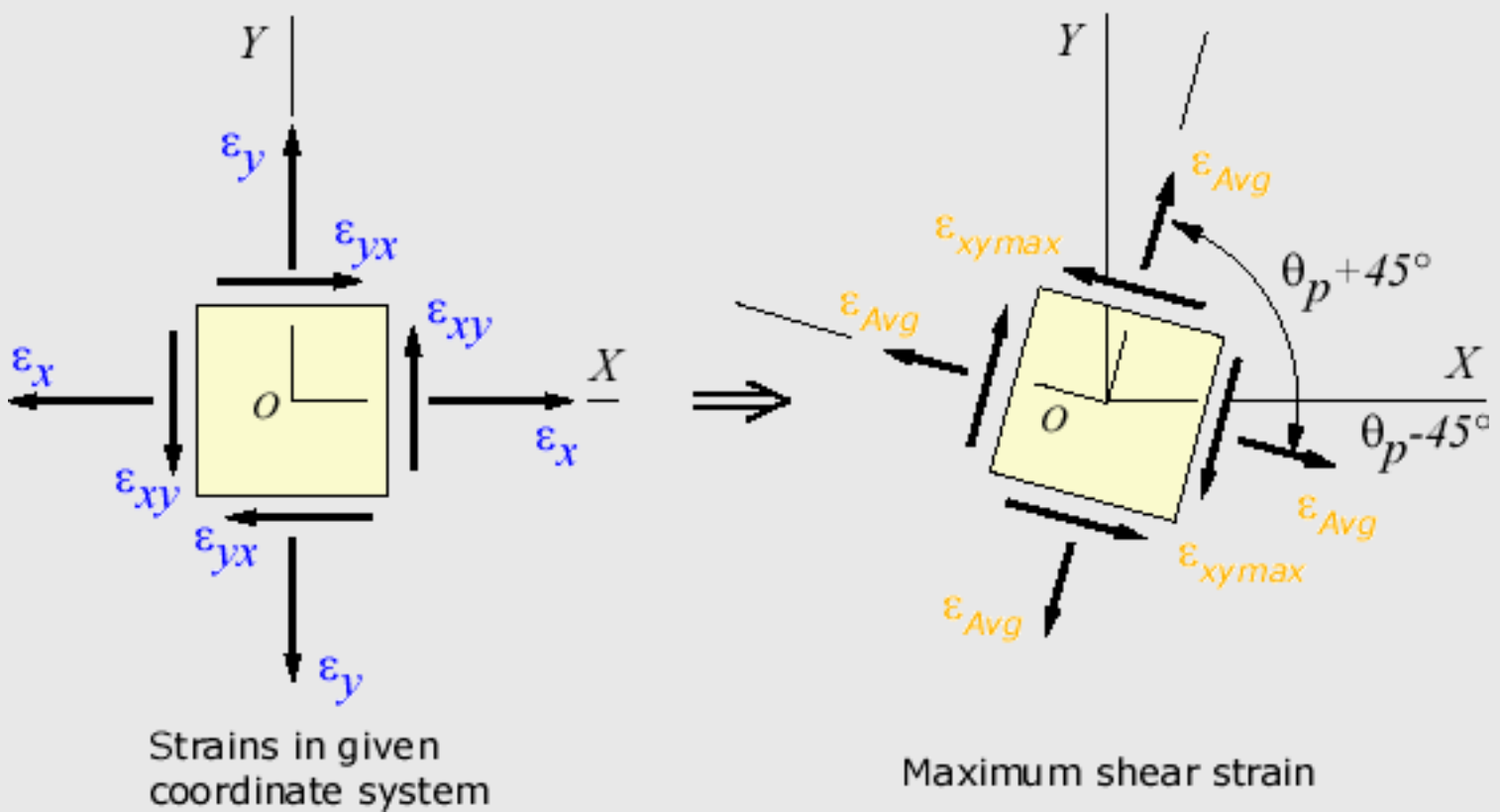
$$\tan 2\theta_s = -\frac{\varepsilon_x - \varepsilon_y}{2\varepsilon_{xy}}$$

$$\Rightarrow \theta_s = \theta_p \pm 45^\circ$$

The maximum shear strain is found to be one-half the difference between the two principal strains,

$$\varepsilon_{\max} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2} = \frac{\varepsilon_1 - \varepsilon_2}{2}$$

The transformation to the maximum shear strain direction can be illustrated as:

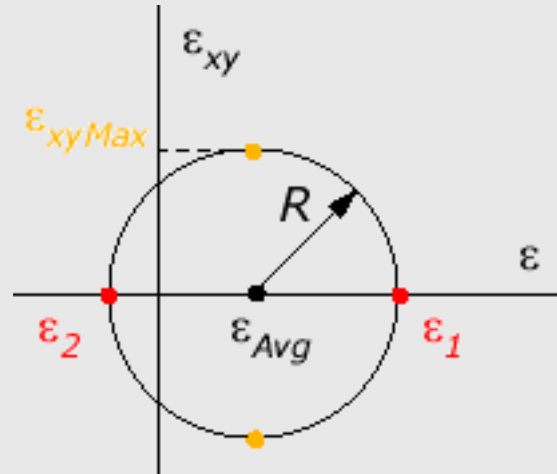


Solid Mechanics: Strain

Mohr's Circle for Plane Strain

Mohr's Circle

Strains at a point in the body can be illustrated by Mohr's Circle. The idea and procedures are exactly the same as for [Mohr's Circle for plane stress](#).



The two principal strains are shown in **red**, and the maximum shear strain is shown in **orange**. Recall that the normal strains are equal to the principal strains when the element is aligned with the principal directions, and the shear strain is equal to the maximum shear strain when the element is rotated 45° away from the principal directions.

As the element is rotated away from the [principal](#) (or maximum strain) directions, the normal and shear strain components will always lie on Mohr's Circle.

Derivation of Mohr's Circle

To establish the Mohr's circle, we first recall the [strain transformation formulas](#) for plane strain,

$$\begin{cases} \varepsilon_{x'} - \frac{\varepsilon_x + \varepsilon_y}{2} = \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \varepsilon_{xy} \sin 2\theta \\ \varepsilon_{x'y'} = -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \varepsilon_{xy} \cos 2\theta \end{cases}$$

Using a basic trigonometric relation ($\cos^2 2\theta + \sin^2 2\theta = 1$) to combine the above two formulas we have,

$$\left(\varepsilon_{x'} - \frac{\varepsilon_x + \varepsilon_y}{2} \right)^2 + \varepsilon_{x'y'}^2 = \left(\frac{\varepsilon_x - \varepsilon_y}{2} \right)^2 + \varepsilon_{xy}^2$$

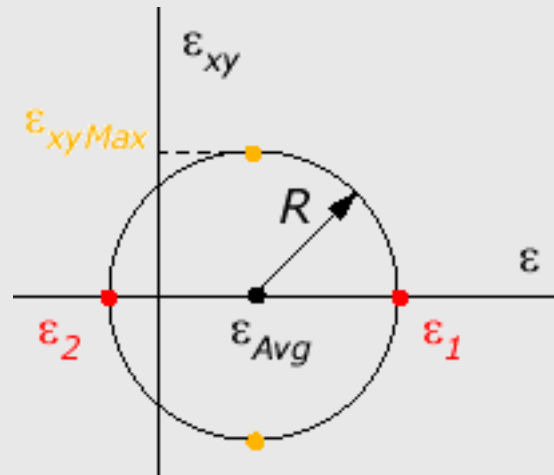
This equation is an equation for a circle. To make this more apparent, we can rewrite it as,

$$(\varepsilon_{x'} - \varepsilon_{Avg})^2 + \varepsilon_{x'y'}^2 = R^2$$

where,

$$\varepsilon_{Avg} = \frac{\varepsilon_x + \varepsilon_y}{2} \quad R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2}$$

The circle is centered at the average strain value ε_{Avg} , and has a radius R equal to the maximum shear strain, as shown in the figure below,



Related Topics

The procedure of drawing Mohr's Circle from a given strain state is discussed in the [Mohr's Circle usage](#) and [examples](#) pages.

The Mohr's Circle for [plane stress](#) can also be obtained from similar procedures.

Solid Mechanics: Strain

Mohr's Circle Usage in Plane Strain

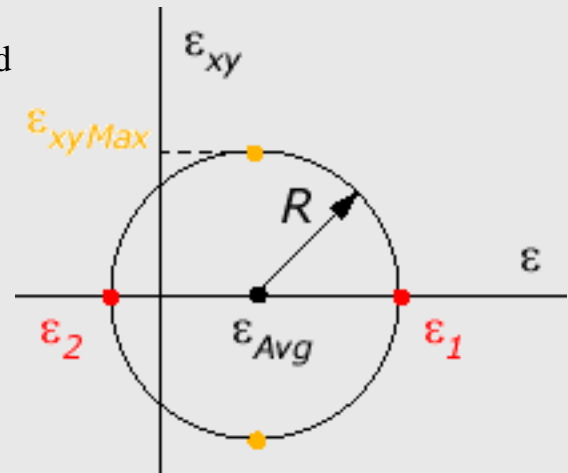
Principal Strains from Mohr's Circle

A chief benefit of Mohr's circle is that the [principal strains](#) ϵ_1 and ϵ_2 and the maximum shear strain ϵ_{xyMax} are obtained immediately after drawing the circle,

$$\begin{cases} \epsilon_{1,2} = \epsilon_{Avg} \pm R \\ \epsilon_{xyMax} = R \end{cases}$$

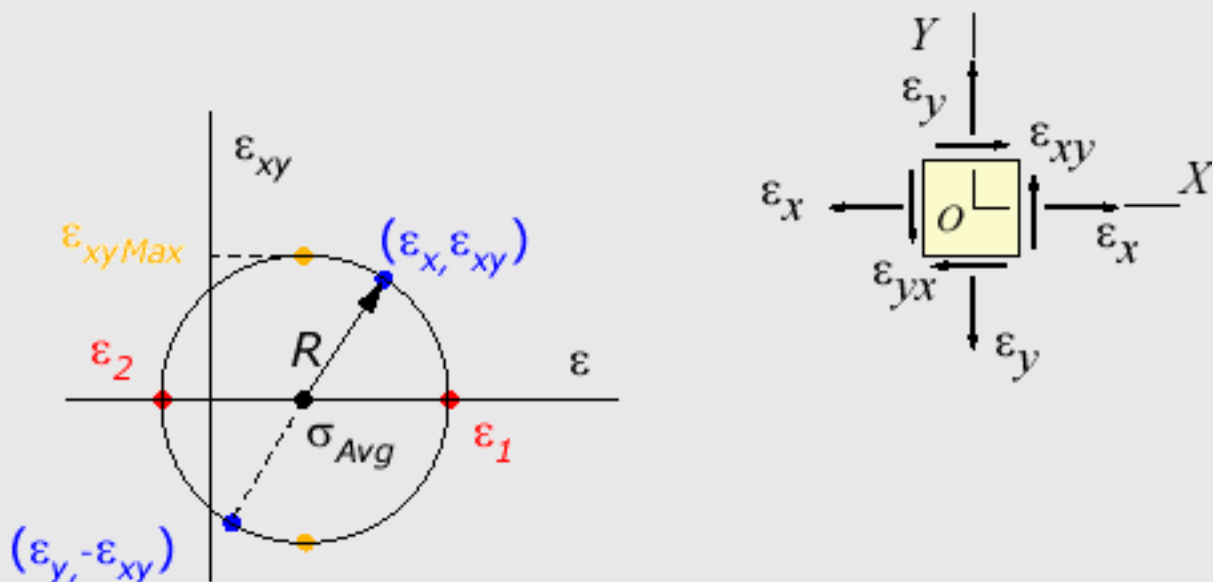
where,

$$\epsilon_{Avg} = \frac{\epsilon_x + \epsilon_y}{2} \quad R = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \epsilon_{xy}^2}$$



Principal Directions from Mohr's Circle

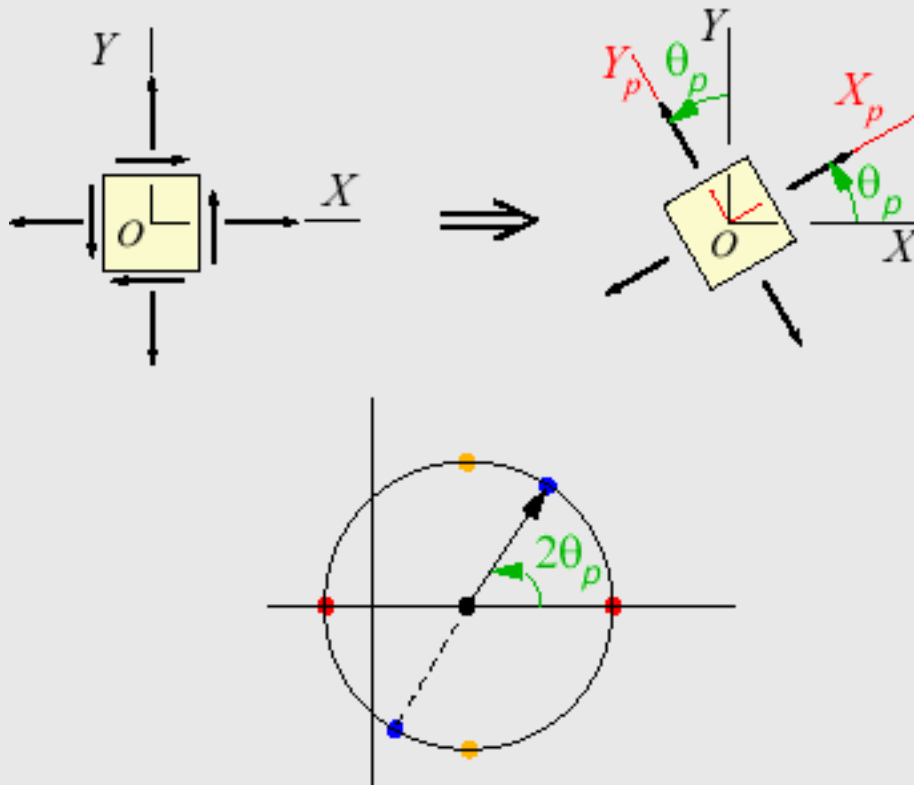
Mohr's Circle can be used to find the directions of the principal axes. To show this, first suppose that the normal and shear strains, ϵ_x , ϵ_y , and ϵ_{xy} , are obtained at a given point O in the body. They are expressed relative to the coordinates XY , as shown in the strain element at right below.



The Mohr's Circle for this general strain state is shown at left above. Note that it's centered at ϵ_{Avg} and has a radius R , and that the two points $(\epsilon_x, \epsilon_{xy})$ and $(\epsilon_y, -\epsilon_{xy})$ lie on opposite sides of the circle. The line connecting ϵ_x and ϵ_y will be defined as L_{xy} .

The **angle** between the current axes (X and Y) and the **principal axes** is defined as θ_p , and is equal to

one half the angle between the line L_{xy} and the ϵ -axis as shown in the schematic below,



A set of six Mohr's Circles representing most strain state possibilities are presented on the [examples](#) page.

Rotation Angle on Mohr's Circle

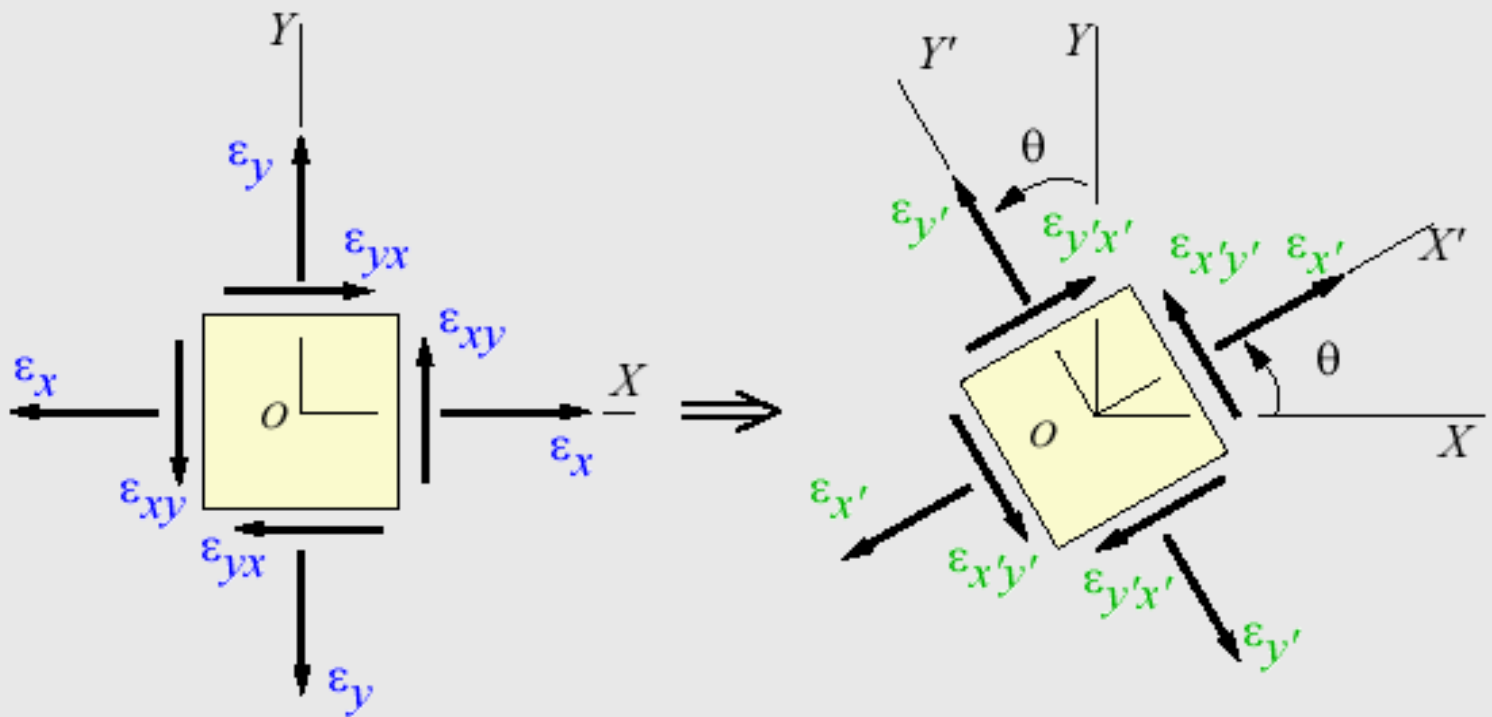
Note that the coordinate rotation angle θ_p is defined positive when starting at the XY coordinates and proceeding to the X_pY_p coordinates. In contrast, on the Mohr's Circle θ_p is defined positive starting on the principal strain line (i.e. the ϵ -axis) and proceeding to the XY strain line (i.e. line L_{xy}). The angle θ_p has the opposite sense between the two figures, because on one it starts on the XY coordinates, and on the other it starts on the principal coordinates.

Some books avoid the sign difference between θ_p on Mohr's Circle and θ_p on the stress element by locating $(\epsilon_x, -\epsilon_{xy})$ instead of $(\epsilon_x, \epsilon_{xy})$ on Mohr's Circle. This will switch the polarity of θ_p on the circle. Whatever method you choose, the bottom line is that an *opposite* sign is needed either in the interpretation or in the plotting to make Mohr's Circle physically meaningful.

Strain Transform by Mohr's Circle

Mohr's Circle can be used to transform strains from one coordinate set to another, similar that that described on the [plane strain](#) page.

Suppose that the normal and shear strains, ϵ_x , ϵ_y , and ϵ_{xy} , are obtained at a point O in the body, expressed with respect to the coordinates XY . We wish to find the strains expressed in the new coordinate set $X'Y'$, rotated an angle θ from XY , as shown below:

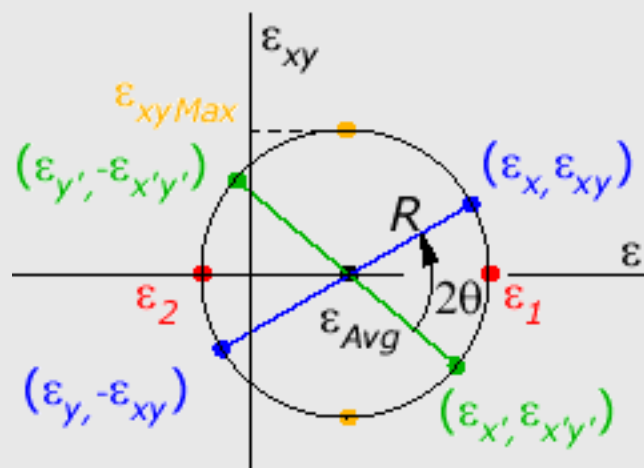


Strains at the given coordinate

Strains transformed to another coordinate

To do this we proceed as follows:

- Draw Mohr's circle for the **given strain state** (ϵ_x , ϵ_y , and ϵ_{xy} ; shown below).
- Draw the line L_{xy} across the circle from $(\epsilon_x, \epsilon_{xy})$ to $(\epsilon_y, -\epsilon_{xy})$.
- Rotate the line L_{xy} by $2*\theta$ (twice as much as the angle between XY and $X'Y'$) and in the *opposite* direction of θ .
- The **strains in the new coordinates** ($\epsilon_{x'}$, $\epsilon_{y'}$, and $\epsilon_{x'y'}$) are then read off the circle.

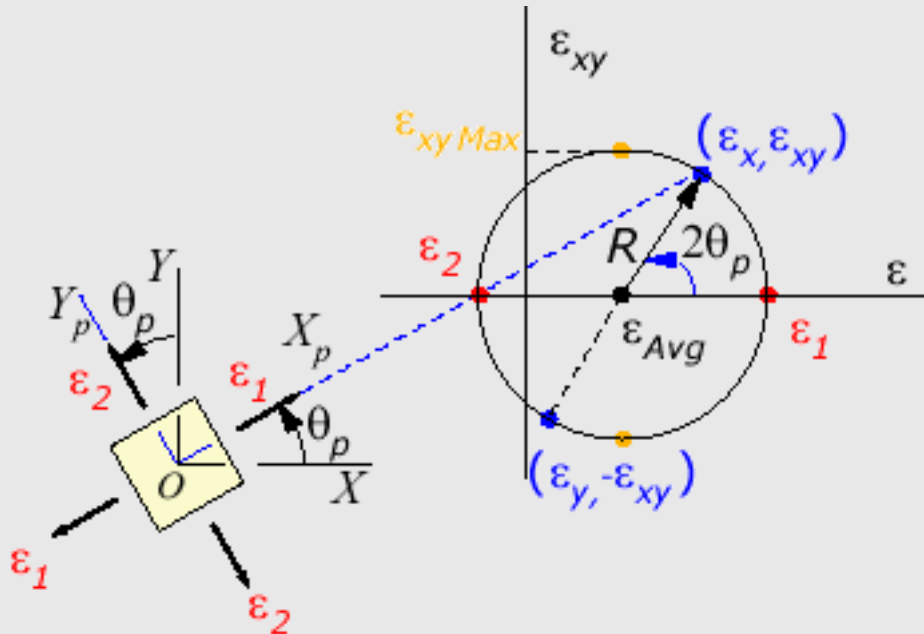


Solid Mechanics: Strain

Examples of Mohr's Circles in Plane Strain

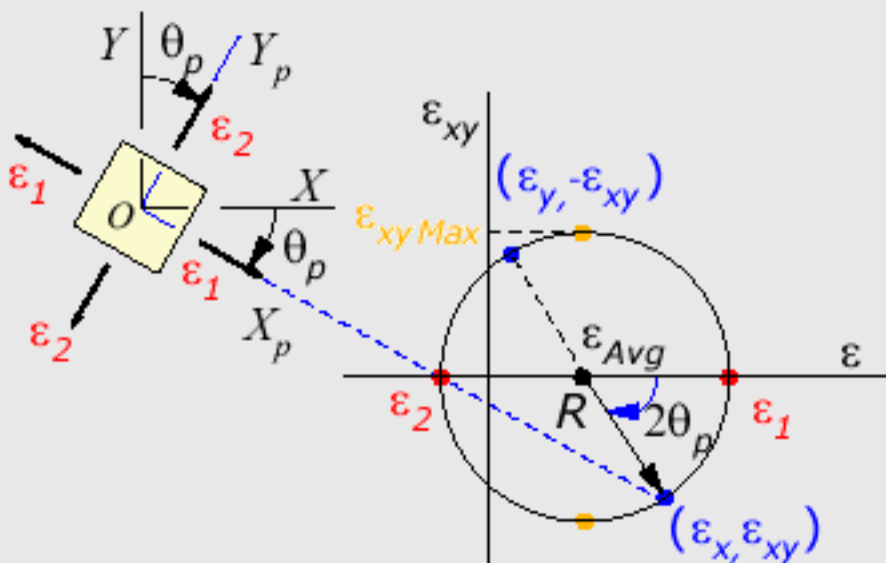
Case 1: $\epsilon_{xy} > 0$ and $\epsilon_x > \epsilon_y$

The principal axes are counterclockwise to the current axes (because $\epsilon_{xy} > 0$) and no more than 45° away (because $\epsilon_x > \epsilon_y$).



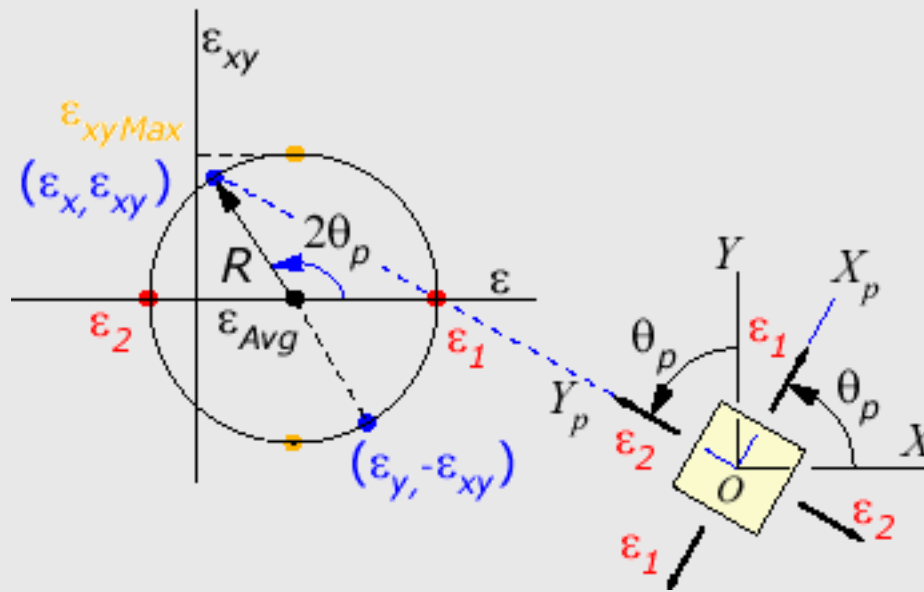
Case 2: $\epsilon_{xy} < 0$ and $\epsilon_x > \epsilon_y$

The principal axes are clockwise to the current axes (because $\epsilon_{xy} < 0$) and no more than 45° away (because $\epsilon_x > \epsilon_y$).

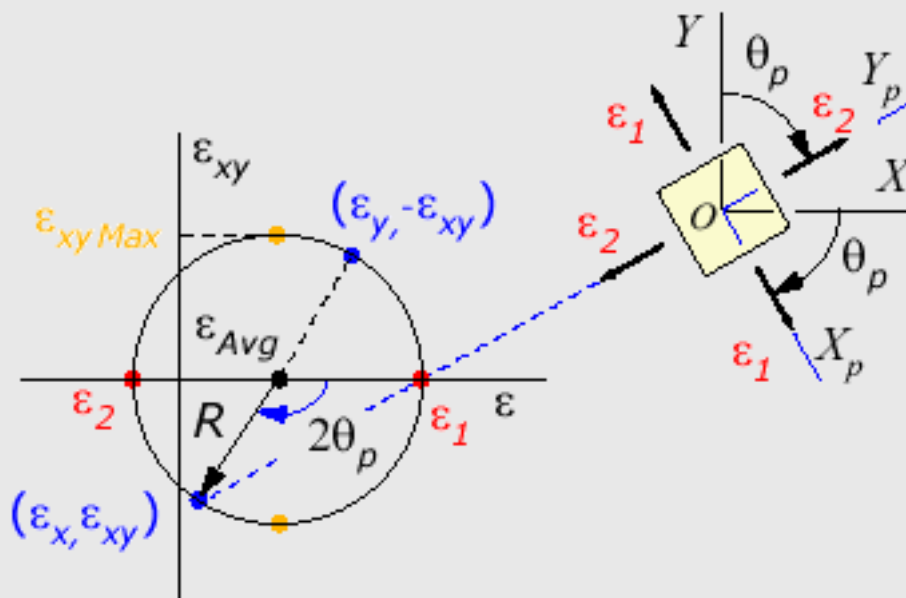


Case 3: $\epsilon_{xy} > 0$ and $\epsilon_x < \epsilon_y$

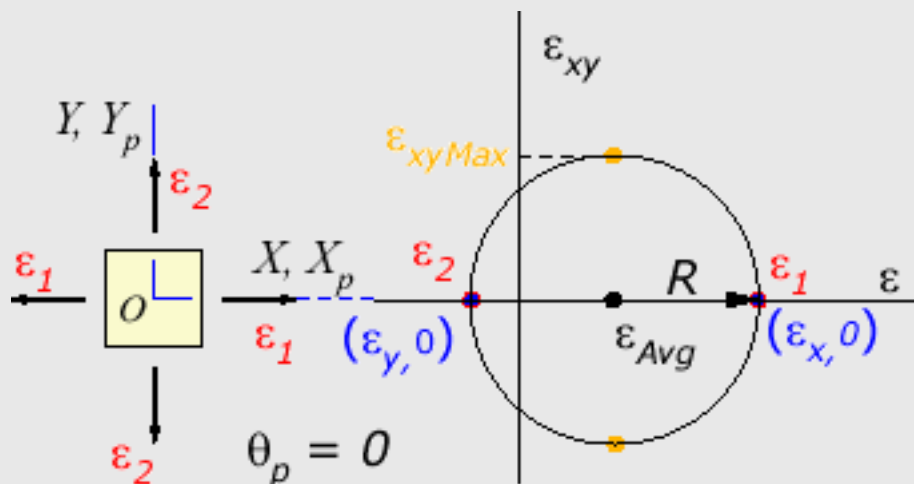
The principal axes are counterclockwise to the current axes (because $\epsilon_{xy} > 0$) and between 45° and 90° away (because $\epsilon_x < \epsilon_y$).

**Case 4: $\epsilon_{xy} < 0$ and $\epsilon_x < \epsilon_y$**

The principal axes are clockwise to the current axes (because $\epsilon_{xy} < 0$) and between 45° and 90° away (because $\epsilon_x < \epsilon_y$).

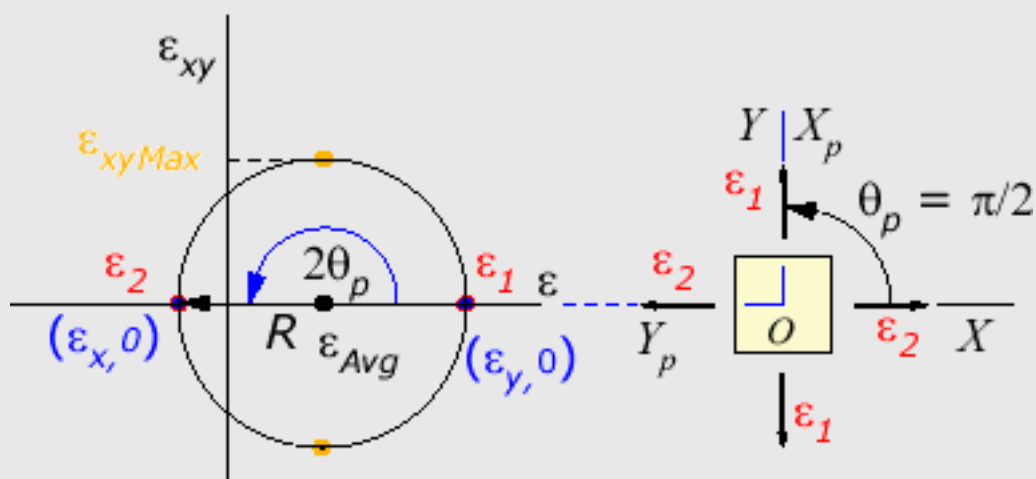
**Case 5: $\epsilon_{xy} = 0$ and $\epsilon_x > \epsilon_y$**

The principal axes are aligned with the current axes (because $\epsilon_x > \epsilon_y$ and $\epsilon_{xy} = 0$).



Case 6: $\epsilon_{xy} = 0$ and $\epsilon_x < \epsilon_y$

The principal axes are exactly 90° from the current axes (because $\epsilon_x < \epsilon_y$ and $\epsilon_{xy} = 0$).



Solid Mechanics: Hooke's Law

Introduction

One-dimensional Hooke's Law

Robert Hooke, who in 1676 stated,

The power (*sic.*) of any springy body is in the same proportion with the extension.

announced the birth of elasticity. Hooke's statement expressed mathematically is,

$$F = k \cdot u$$

where F is the applied force (and not the power, as Hooke mistakenly suggested), u is the deformation of the elastic body subjected to the force F , and k is the spring constant (i.e. the ratio of previous two parameters).

Generalized Hooke's Law (Anisotropic Form)

Cauchy generalized Hooke's law to three dimensional elastic bodies and stated that the 6 components of stress are linearly related to the 6 components of strain.

The stress-strain relationship written in matrix form, where the 6 components of [stress](#) and [strain](#) are organized into column vectors, is,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}, \quad \varepsilon = \mathbf{S} \cdot \sigma$$

or,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix}, \quad \sigma = \mathbf{C} \cdot \varepsilon$$

where **C** is the **stiffness matrix**, **S** is the **compliance matrix**, and $\mathbf{S} = \mathbf{C}^{-1}$.

In general, stress-strain relationships such as these are known as **constitutive relations**.

In general, there are 36 stiffness matrix components. However, it can be shown that conservative materials possess a strain energy density function and as a result, the stiffness and compliance matrices are symmetric. Therefore, only 21 stiffness components are actually independent in Hooke's law. The vast majority of engineering materials are conservative.

Please note that the **stiffness** matrix is traditionally represented by the symbol **C**, while **S** is reserved for the **compliance** matrix. This convention may seem backwards, but perception is not always reality. For instance, Americans hardly ever use their feet to play (American) football.

Solid Mechanics: Hooke's Law

Hooke's Law for Orthotropic Materials

Orthotropic Definition

Some engineering materials, including certain piezoelectric materials (e.g. Rochelle salt) and 2-ply fiber-reinforced composites, are **orthotropic**.

By definition, an orthotropic material has at least 2 orthogonal planes of symmetry, where material properties are independent of direction within each plane. Such materials require 9 independent variables (i.e. elastic constants) in their constitutive matrices.

In contrast, a material without any planes of symmetry is fully [anisotropic](#) and requires 21 elastic constants, whereas a material with an infinite number of symmetry planes (i.e. every plane is a plane of symmetry) is [isotropic](#), and requires only 2 elastic constants.

Hooke's Law in Compliance Form

By convention, the 9 elastic constants in orthotropic constitutive equations are comprised of 3 Young's moduli E_x, E_y, E_z , the 3 Poisson's ratios $\nu_{yz}, \nu_{zx}, \nu_{xy}$, and the 3 shear moduli G_{yz}, G_{zx}, G_{xy} .

The **compliance matrix** takes the form,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{zx}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}$$

where $\frac{\nu_{yz}}{E_y} = \frac{\nu_{zy}}{E_z}$, $\frac{\nu_{zx}}{E_z} = \frac{\nu_{xz}}{E_x}$, $\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y}$.

Note that, in orthotropic materials, there is no interaction between the normal stresses σ_x , σ_y , σ_z and the shear strains ϵ_{yz} , ϵ_{zx} , ϵ_{xy}

The factor 2 multiplying the shear moduli in the compliance matrix results from the difference between shear strain and [engineering shear strain](#), where $\gamma_{xy} = \epsilon_{xy} + \epsilon_{yx} = 2\epsilon_{xy}$, etc.

Hooke's Law in Stiffness Form

The **stiffness matrix** for orthotropic materials, found from the inverse of the compliance matrix, is given by,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1 - \nu_{yz}\nu_{zy}}{E_y E_z \Delta} & \frac{\nu_{yx} + \nu_{zx}\nu_{yz}}{E_y E_z \Delta} & \frac{\nu_{zx} + \nu_{yx}\nu_{zy}}{E_y E_z \Delta} & 0 & 0 & 0 \\ \frac{\nu_{xy} + \nu_{xz}\nu_{zx}}{E_z E_x \Delta} & \frac{1 - \nu_{zx}\nu_{xz}}{E_z E_x \Delta} & \frac{\nu_{zy} + \nu_{zx}\nu_{xy}}{E_z E_x \Delta} & 0 & 0 & 0 \\ \frac{\nu_{xz} + \nu_{xy}\nu_{yz}}{E_x E_y \Delta} & \frac{\nu_{zy} + \nu_{xz}\nu_{yz}}{E_x E_y \Delta} & \frac{1 - \nu_{xy}\nu_{yx}}{E_x E_y \Delta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G_{yz} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G_{zx} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G_{xy} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{zx} \\ \epsilon_{xy} \end{bmatrix}$$

where,

$$\Delta = \frac{1 - \nu_{xy}\nu_{yx} - \nu_{yz}\nu_{zy} - \nu_{zx}\nu_{xz} - \nu_{xy}\nu_{yz}\nu_{zx}}{E_x E_y E_z}$$

The fact that the stiffness matrix is symmetric requires that the following statements hold,

$$\begin{cases} \frac{\nu_{yx} + \nu_{zx}\nu_{yz}}{E_y E_z \Delta} = \frac{\nu_{xy} + \nu_{xz}\nu_{zx}}{E_z E_x \Delta} \\ \frac{\nu_{zy} + \nu_{zx}\nu_{xy}}{E_z E_x \Delta} = \frac{\nu_{zy} + \nu_{xz}\nu_{yz}}{E_x E_y \Delta} \\ \frac{\nu_{zx} + \nu_{yx}\nu_{zy}}{E_y E_z \Delta} = \frac{\nu_{xz} + \nu_{xy}\nu_{yz}}{E_x E_y \Delta} \end{cases}$$

The factor of 2 multiplying the shear moduli in the stiffness matrix results from the difference between shear strain and [engineering shear strain](#), where $\gamma_{xy} = \epsilon_{xy} + \epsilon_{yx} = 2\epsilon_{xy}$, etc.

Solid Mechanics: Hooke's Law

Hooke's Law for Transversely Isotropic Materials

Transverse Isotropic Definition

A special class of [orthotropic](#) materials are those that have the same properties in one plane (e.g. the x - y plane) and different properties in the direction normal to this plane (e.g. the z -axis). Such materials are called **transverse isotropic**, and they are described by 5 independent elastic constants, instead of 9 for fully orthotropic.

Examples of transversely isotropic materials include some piezoelectric materials (e.g. PZT-4, barium titanate) and fiber-reinforced composites where all fibers are in parallel.

Hooke's Law in Compliance Form

By convention, the 5 elastic constants in transverse isotropic constitutive equations are the Young's modulus and poisson ratio in the x - y symmetry plane, E_p and ν_p , the Young's modulus and poisson ratio in the z -direction, E_{pz} and ν_{pz} , and the shear modulus in the z -direction G_{zp} .

The **compliance matrix** takes the form,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_p} & -\frac{\nu_p}{E_p} & -\frac{\nu_{zp}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_p}{E_p} & \frac{1}{E_p} & -\frac{\nu_{zp}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{pz}}{E_p} & -\frac{\nu_{pz}}{E_p} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{zp}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{zp}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+\nu_p}{E_p} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}$$

where $\frac{\nu_{pz}}{E_p} = \frac{\nu_{zp}}{E_z}$.

The factor 2 multiplying the shear moduli in the compliance matrix results from the difference

between shear strain and [engineering shear strain](#), where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$, etc.

Hooke's Law in Stiffness Form

The **stiffness matrix** for transverse isotropic materials, found from the inverse of the compliance matrix, is given by,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1 - \nu_{pz}\nu_{zp}}{E_p E_z \Delta} & \frac{\nu_p + \nu_{zp}\nu_{pz}}{E_p E_z \Delta} & \frac{\nu_{zp} + \nu_p\nu_{zp}}{E_p E_z \Delta} & 0 & 0 & 0 \\ \frac{\nu_p + \nu_{pz}\nu_{zp}}{E_z E_p \Delta} & \frac{1 - \nu_{zp}\nu_{pz}}{E_z E_p \Delta} & \frac{\nu_{zp} + \nu_{zp}\nu_p}{E_z E_p \Delta} & 0 & 0 & 0 \\ \frac{\nu_{pz} + \nu_p\nu_{pz}}{E_p^2 \Delta} & \frac{\nu_{zp} + \nu_{pz}^2}{E_p^2 \Delta} & \frac{1 - \nu_p^2}{E_p^2 \Delta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G_{zp} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G_{zp} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{E_p}{1 + \nu_p} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix}$$

where,

$$\Delta = \frac{1 - \nu_p^2 - 2\nu_{pz}\nu_{zp} - \nu_p\nu_{pz}\nu_{zp}}{E_p^2 E_z}$$

The fact that the stiffness matrix is symmetric requires that the following statements hold,

$$\begin{cases} \frac{\nu_p + \nu_{zp}\nu_{pz}}{E_p E_z \Delta} = \frac{\nu_p + \nu_{pz}\nu_{zp}}{E_z E_p \Delta} \\ \frac{\nu_{zp} + \nu_{zp}\nu_p}{E_z E_p \Delta} = \frac{\nu_{zp} + \nu_{pz}^2}{E_p^2 \Delta} \\ \frac{\nu_{zp} + \nu_p\nu_{zp}}{E_p E_z \Delta} = \frac{\nu_{pz} + \nu_p\nu_{pz}}{E_p^2 \Delta} \end{cases}$$

The factor of 2 multiplying the shear moduli in the stiffness matrix results from the difference between shear strain and [engineering shear strain](#), where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$, etc.

Solid Mechanics: Hooke's Law

Hooke's Law for Isotropic Materials

Isotropic Definition

Most metallic alloys and thermoset polymers are considered **isotropic**, where by definition the material properties are independent of direction. Such materials have only 2 independent variables (i.e. elastic constants) in their stiffness and compliance matrices, as opposed to the 21 elastic constants in the general [anisotropic](#) case.

The two elastic constants are usually expressed as the [Young's modulus](#) E and the [Poisson's ratio](#) ν . However, the alternative elastic constants K ([bulk modulus](#)) and/or G ([shear modulus](#)) can also be used. For isotropic materials, G and K can be found from E and ν by a set of equations, and vice-versa.

Hooke's Law in Compliance Form

Hooke's law for isotropic materials in **compliance matrix** form is given by,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}$$

Hooke's Law in Stiffness Form

The **stiffness matrix** is equal to the inverse of the compliance matrix, and is given by,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix}$$

Solid Mechanics: Hooke's Law

Hooke's Law for Plane Stress

Hooke's Law for Plane Stress

For the simplification of [plane stress](#), where the stresses in the z direction are considered to be negligible, $\sigma_{zz} = \sigma_{yz} = \sigma_{xz} = 0$, the stress-strain compliance relationship for an [isotropic](#) material becomes,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ 0 \\ 0 \\ 0 \\ \sigma_{xy} \end{bmatrix}$$

The three zero'd stress entries in the stress vector indicate that we can ignore their associated columns in the compliance matrix (i.e. columns 3, 4, and 5). If we also ignore the rows associated with the strain components with z -subscripts, the **compliance matrix** reduces to a simple 3x3 matrix,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

The **stiffness matrix** for plane stress is found by inverting the plane stress compliance matrix, and is given by,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}$$

Note that the stiffness matrix for plane stress is **NOT** found by removing columns and rows from the general [isotropic stiffness matrix](#).

Plane Stress Hooke's Law via Engineering Strain

Some reference books incorporate the shear modulus G and the [engineering shear strain](#) γ_{xy} , related to the shear strain ε_{xy} via,

$$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$$

The stress-strain **compliance matrix** using G and γ_{xy} are,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

The **stiffness matrix** is,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

The shear modulus G is related to E and ν via,

$$G = \frac{E}{2(1+\nu)}$$

Solid Mechanics: Hooke's Law

Hooke's Law for Plane Strain

Hooke's Law for Plane Strain

For the case of [plane strain](#), where the strains in the z direction are considered to be negligible, $\varepsilon_{zz} = \varepsilon_{yz} = \varepsilon_{xz} = 0$, the stress-strain stiffness relationship for an [isotropic](#) material becomes,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 0 \\ 0 \\ 0 \\ \varepsilon_{xy} \end{bmatrix}$$

The three zero'd strain entries in the strain vector indicate that we can ignore their associated columns in the stiffness matrix (i.e. columns 3, 4, and 5). If we also ignore the rows associated with the stress components with z -subscripts, the **stiffness matrix** reduces to a simple 3x3 matrix,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

The **compliance matrix** for plane stress is found by inverting the plane stress stiffness matrix, and is given by,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

Note that the compliance matrix for plane stress is **NOT** found by removing columns and rows from the general [isotropic compliance matrix](#).

Plane Strain Hooke's Law via Engineering Strain

The stress-strain **stiffness matrix** expressed using the shear modulus G and the engineering shear strain $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$ is,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 \\ \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

The **compliance matrix** is,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1-\nu^2}{E} & -\frac{\nu(1+\nu)}{E} & 0 \\ -\frac{\nu(1+\nu)}{E} & \frac{1-\nu^2}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

The shear modulus G is related to E and ν via,

$$G = \frac{E}{2(1+\nu)}$$

Solid Mechanics: Hooke's Law

Finding Young's Modulus and Poisson's Ratio

Young's Modulus from Uniaxial Tension

When a specimen made from an [isotropic](#) material is subjected to uniaxial tension, say in the x direction, σ_{xx} is the only non-zero stress. The strains in the specimen are obtained by,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The modulus of elasticity in tension, also known as **Young's modulus** E , is the ratio of stress to strain on the loading plane along the loading direction,

$$E = \frac{\sigma_{xx}}{\varepsilon_{xx}}$$

Common sense (and the 2nd Law of Thermodynamics) indicates that a material under uniaxial tension must elongate in length. Therefore the Young's modulus E is required to be non-negative for all materials,

$$E > 0$$

Poisson's Ratio from Uniaxial Tension

A rod-like specimen subjected to uniaxial tension will exhibit some shrinkage in the lateral direction for most materials. The ratio of lateral strain and axial strain is defined as **Poisson's ratio** ν ,

$$\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}}$$

The Poisson ratio for most metals falls between 0.25 to 0.35. Rubber has a Poisson ratio close to 0.5 and is therefore almost incompressible. Theoretical materials with a Poisson ratio of **exactly 0.5** are truly **incompressible**, since the sum of all their strains leads to a zero volume change. Cork, on the other hand, has a Poisson ratio close to zero. This makes cork function well as a bottle stopper, since an axially-loaded cork will not swell laterally to resist bottle insertion.

The Poisson's ratio is bounded by two theoretical limits: it must be [greater than -1](#), and [less than or](#)

[equal to 0.5](#),

$$-1 < \nu \leq \frac{1}{2}$$

The [proof](#) for this stems from the fact that E , G , and K are all positive and mutually dependent. However, it is rare to encounter engineering materials with negative Poisson ratios. Most materials will fall in the range,

$$0 \leq \nu \leq \frac{1}{2}$$

Solid Mechanics: Hooke's Law

Finding the Shear Modulus and the Bulk Modulus

Shear Modulus from Pure Shear

When a specimen made from an [isotropic](#) material is subjected to pure shear, for instance, a cylindrical bar under torsion in the xy sense, σ_{xy} is the only non-zero stress. The strains in the specimen are obtained by,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sigma_{xy} \end{bmatrix}$$

The **shear modulus** G , is defined as the ratio of shear stress to [engineering shear strain](#) on the loading plane,

$$\begin{aligned} G &= \frac{\sigma_{xy}}{\varepsilon_{xy} + \varepsilon_{yx}} = \frac{\sigma_{xy}}{2\varepsilon_{xy}} = \frac{\sigma_{xy}}{\gamma_{xy}} \\ &= \frac{E}{2(1+\nu)} \end{aligned}$$

where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$.

The shear modulus G is also known as the rigidity modulus, and is equivalent to the 2nd Lamé constant μ mentioned in books on continuum theory.

Common sense and the 2nd Law of Thermodynamics require that a positive shear stress leads to a positive shear strain. Therefore, the shear modulus G is required to be nonnegative for all materials,

$$G > 0$$

Since both G and the elastic modulus E are required to be positive, the quantity in the denominator of G must also be positive. This requirement places a **lower bound restriction on the range for Poisson's ratio**,

$$\nu > -1$$

Bulk Modulus from Hydrostatic Pressure

When an [isotropic](#) material specimen is subjected to hydrostatic pressure σ , all shear stress will be zero and the normal stress will be uniform, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma$. The strains in the specimen are given by,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma \\ \sigma \\ \sigma \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In response to the hydrostatic load, the specimen will change its volume. Its resistance to do so is quantified as the **bulk modulus K** , also known as the modulus of compression. Technically, K is defined as the ratio of hydrostatic pressure to the [relative volume change](#) (which is related to the direct strains),

$$\begin{aligned} K &= \frac{\sigma}{\Delta V/V} = \frac{\sigma}{\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}} \\ &= \frac{E}{3(1-2\nu)} \end{aligned}$$

Common sense and the 2nd Law of Thermodynamics require that a positive hydrostatic load leads to a positive volume change. Therefore, the bulk modulus K is required to be nonnegative for all materials,

$$K > 0$$

Since both K and the elastic modulus E are required to be positive, the following requirement is placed on the **upper bound of Poisson's ratio** by the denominator of K ,

$$\nu < 1/2$$

Relation Between Relative Volume Change and Strain

For simplicity, consider a rectangular block of material with dimensions a_0 , b_0 , and c_0 . Its volume V_0 is given by,

$$V_0 = a_0 b_0 c_0$$

When the block is loaded by stress, its volume will change since each dimension now includes a direct strain measure. To calculate the volume when loaded V_f we multiply the new dimensions of the block,

$$\begin{aligned} V_f &= a_f b_f c_f = [a_0 (1 + \varepsilon_{xx})][b_0 (1 + \varepsilon_{yy})][c_0 (1 + \varepsilon_{zz})] \\ &= V_0 (1 + \varepsilon_{xx})(1 + \varepsilon_{yy})(1 + \varepsilon_{zz}) \\ &= V_0 (1 + \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx} + \varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz}) \\ &\approx V_0 (1 + \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) \end{aligned}$$

Products of strain measures will be much smaller than individual strain measures when the overall strain in the block is small (i.e. linear strain theory). Therefore, we were able to drop the strain products in the equation above.

The relative change in volume is found by dividing the volume difference by the initial volume,

$$\frac{\Delta V}{V_0} = \frac{V_f - V_0}{V_0} \approx \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

Hence, the relative volume change (for small strains) is equal to the sum of the 3 direct strains.

Solid Mechanics: Failure Criteria

Introduction

Stress-Based Criteria

The purpose of **failure criteria** is to predict or estimate the failure/yield of machine parts and structural members.

A considerable number of theories have been proposed. However, only the most common and well-tested theories applicable to [isotropic](#) materials are discussed here. These theories, dependent on the nature of the material in question (i.e. brittle or ductile), are listed in the following table:

Material Type	Failure Theories
Ductile	Maximum shear stress criterion , von Mises criterion
Brittle	Maximum normal stress criterion , Mohr's theory

All four criteria are presented in terms of [principal stresses](#). Therefore, all stresses should be [transformed](#) to the principal stresses before applying these failure criteria.

- Note: 1. Whether a material is *brittle* or *ductile* could be a subjective guess, and often depends on temperature, strain levels, and other environmental conditions. However, a 5% *elongation* criterion at break is a reasonable dividing line. Materials with a larger elongation can be considered ductile and those with a lower value brittle. Another distinction is a brittle material's compression strength is usually significantly larger than its tensile strength.
2. All popular failure criteria rely on only a handful of basic tests (such as uniaxial tensile and/or compression strength), even though most machine parts and structural members are typically subjected to multi-axial loading. This disparity is usually driven by cost, since complete multi-axial failure testing requires extensive, complicated, and expensive tests.

Non Stress-Based Criteria

The success of all machine parts and structural members are not necessarily determined by their strength. Whether a part succeeds or fails may depend on other factors, such as stiffness, vibrational characteristics, fatigue resistance, and/or creep resistance.

For example, the automobile industry has endeavored many years to increase the rigidity of passenger cages and install additional safety equipment. The bicycle industry continues to decrease the weight and increase the stiffness of bicycles to enhance their performance.

In civil engineering, a patio deck only needs to be strong enough to carry the weight of several people. However, a design based on the "strong enough" precept will often result a bouncy deck that most people will find objectionable. Rather, the *stiffness* of the deck determines the success of the design.

Many factors, in addition to stress, may contribute to the design requirements of a part. Together, these requirements are intended to increase the sense of security, safety, and quality of service of the part.

Solid Mechanics: Failure Criteria

Failure Criteria for Ductile Materials

Maximum Shear Stress Criterion

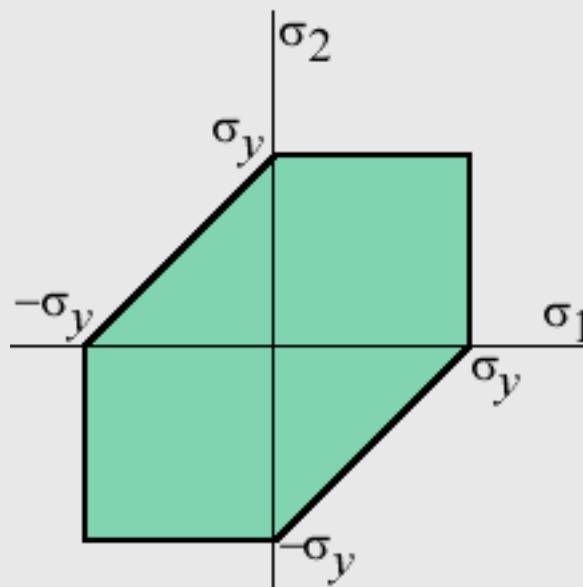
The maximum shear stress criterion, also known as Tresca's or Guest's criterion, is often used to predict the yielding of ductile materials.

Yield in ductile materials is usually caused by the *slippage* of crystal planes along the maximum shear stress surface. Therefore, a given point in the body is considered safe as long as the maximum shear stress at that point is under the yield shear stress σ_y obtained from a uniaxial tensile test.

With respect to 2D stress, the maximum shear stress is related to the difference in the two [principal stresses](#) (see [Mohr's Circle](#)). Therefore, the criterion requires the principal stress difference, along with the principal stresses themselves, to be less than the yield shear stress,

$$|\sigma_1| \leq \sigma_y, \quad |\sigma_2| \leq \sigma_y, \quad \text{and} \quad |\sigma_1 - \sigma_2| \leq \sigma_y$$

Graphically, the maximum shear stress criterion requires that the two principal stresses be within the green zone indicated below,



Von Mises Criterion

The von Mises Criterion (1913), also known as the maximum distortion energy criterion, octahedral shear stress theory, or Maxwell-Huber-Hencky-von Mises theory, is often used to estimate the yield of ductile materials.

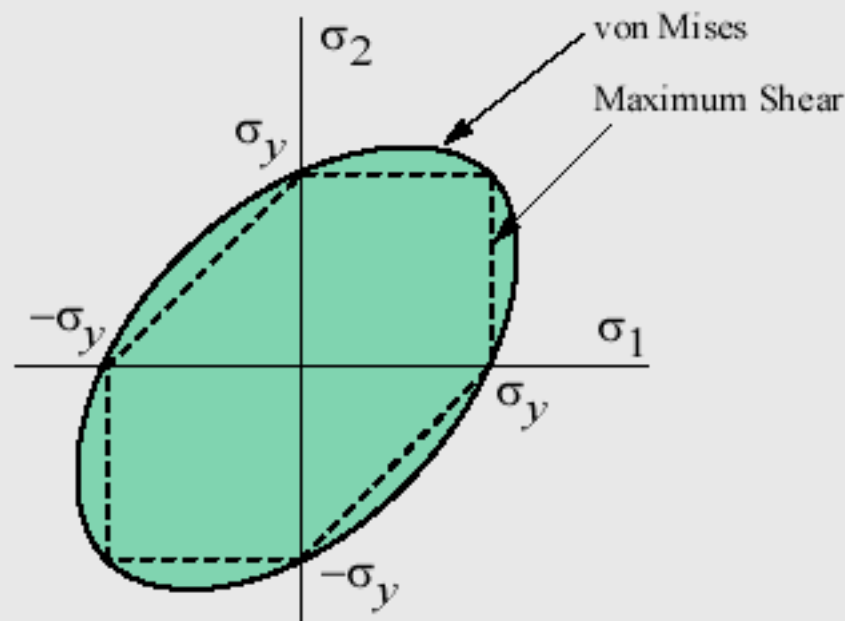
The von Mises criterion states that failure occurs when the energy of distortion reaches the same energy for yield/failure in uniaxial tension. Mathematically, this is expressed as,

$$\frac{1}{2} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \leq \sigma_y^2$$

In the cases of plane stress, $\sigma_3 = 0$. The von Mises criterion reduces to,

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 \leq \sigma_y^2$$

This equation represents a principal stress ellipse as illustrated in the following figure,



Also shown on the figure is the [maximum shear stress criterion](#) (dashed line). This theory is more conservative than the von Mises criterion since it lies inside the von Mises ellipse.

In addition to bounding the principal stresses to prevent ductile failure, the von Mises criterion also gives a reasonable estimation of fatigue failure, especially in cases of repeated tensile and tensile-shear loading.

Solid Mechanics: Failure Criteria

Failure Criteria for Brittle Materials

Maximum Normal Stress Criterion

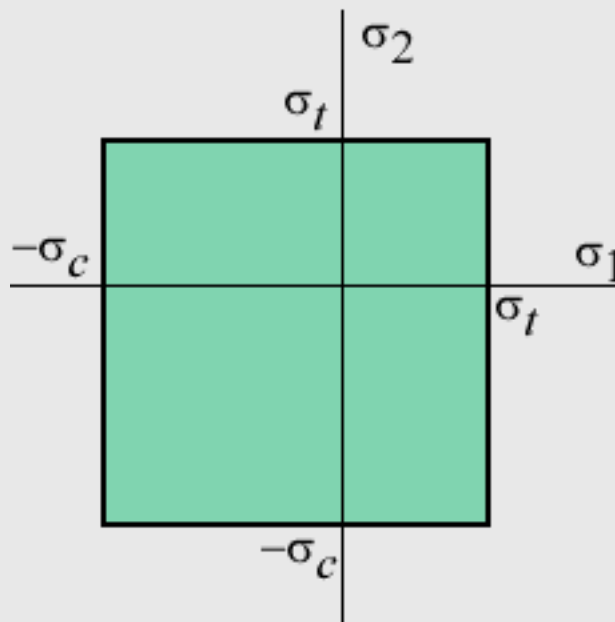
The maximum stress criterion, also known as the normal stress, Coulomb, or Rankine criterion, is often used to predict the failure of brittle materials.

The maximum stress criterion states that failure occurs when the maximum (normal) [principal stress](#) reaches either the *uniaxial* tension strength σ_t , or the *uniaxial* compression strength σ_c ,

$$-\sigma_c < \{\sigma_1, \sigma_2\} < \sigma_t$$

where σ_1 and σ_2 are the principal stresses for 2D stress.

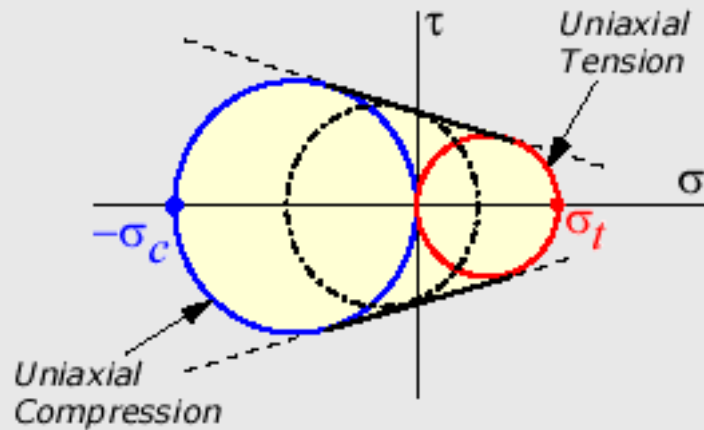
Graphically, the maximum stress criterion requires that the two principal stresses lie within the green zone depicted below,



Mohr's Theory

The Mohr Theory of Failure, also known as the Coulomb-Mohr criterion or internal-friction theory, is based on the famous [Mohr's Circle](#). Mohr's theory is often used in predicting the failure of brittle materials, and is applied to cases of 2D stress.

Mohr's theory suggests that failure occurs when Mohr's Circle at a point in the body exceeds the envelope created by the two Mohr's circles for uniaxial tensile strength and uniaxial compression strength. This envelope is shown in the figure below,



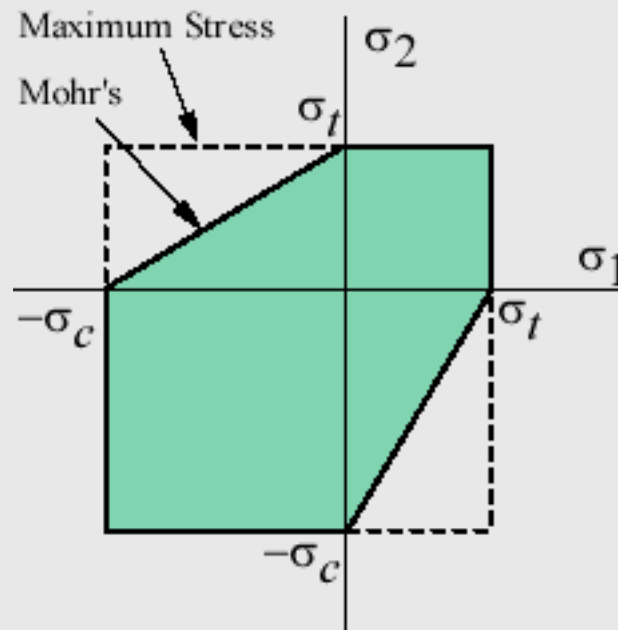
The left circle is for uniaxial compression at the limiting compression stress σ_c of the material. Likewise, the right circle is for uniaxial tension at the limiting tension stress σ_t .

The middle Mohr's Circle on the figure (dash-dot-dash line) represents the maximum allowable stress for an intermediate stress state.

All intermediate stress states fall into one of the four categories in the following table. Each case defines the maximum allowable values for the two principal stresses to avoid failure.

Case	Principal Stresses		Criterion requirements
1	Both in tension	$\sigma_1 > 0, \sigma_2 > 0$	$\sigma_1 < \sigma_t, \sigma_2 < \sigma_t$
2	Both in compression	$\sigma_1 < 0, \sigma_2 < 0$	$\sigma_1 > -\sigma_c, \sigma_2 > -\sigma_c$
3	σ_1 in tension, σ_2 in compression	$\sigma_1 > 0, \sigma_2 < 0$	$\frac{\sigma_1}{\sigma_t} + \frac{\sigma_2}{-\sigma_c} < 1$
4	σ_1 in compression, σ_2 in tension	$\sigma_1 < 0, \sigma_2 > 0$	$\frac{\sigma_1}{-\sigma_c} + \frac{\sigma_2}{\sigma_t} < 1$

Graphically, Mohr's theory requires that the two principal stresses lie within the green zone depicted below,



Also shown on the figure is the [maximum stress criterion](#) (dashed line). This theory is less conservative than Mohr's theory since it lies outside Mohr's boundary.

Solid Mechanics: Failure Criteria

Techniques for Failure Prevention and Diagnosis

There exist a set of basic techniques for preventing failure in the design stage, and for diagnosing failure in manufacturing and later stages.

In the Design Stage

It is quite commonplace today for design engineers to verify design stresses with finite element (FEA) packages. This is fine and good when FEA is applied appropriately. However, the popularity of finite element analysis can condition engineers to look just for red spots in simulation output, without really understanding the essence or *funda* at play.

By following basic rules of thumb, such danger points can often be anticipated and avoided without total reliance on computer simulation.

Loading Points	Maximum stresses are often located at loading points, supports, joints, or maximum deflection points.
Stress Concentrations	<p>Stress concentrations are usually located near corners, holes, crack tips, boundaries, between layers, and where cross-section areas change rapidly.</p> <p>Sound design avoids rapid changes in material or geometrical properties. For example, when a large hole is removed from a structure, a reinforcement composed of generally no less than the material removed should be added around the opening.</p>
Safety Factors	The addition of safety factors to designs allow engineers to reduce sensitivity to manufacturing defects and to compensate for stress prediction limitations.

In Post-Manufacturing Stages

Despite the best efforts of design and manufacturing engineers, unanticipated failure may occur in parts after design and manufacturing. In order for projects to succeed, these failures must be diagnosed and resolved quickly and effectively. Often, the failure is caused by a singular factor, rather than an involved collection of factors.

Such failures may be caught early in initial quality assurance testing, or later after the part is delivered to the customer.

Induced Stress Concentrations	<p>Stress concentrations may be induced by inadequate manufacturing processes.</p> <p>For example, initial surface imperfections can result from sloppy machining processes. Manufacturing defects such as size mismatches and improper fastener application can lead to residual stresses and even cracks, both strong stress concentrations.</p>
Damage and Exposure	<p>Damages during service life can lead a part to failure. Damages such as cracks, debonding, and delamination can result from unanticipated resonant vibrations and impacts that exceed the design loads.</p> <p>Reduction in strength can result from exposure to UV lights and chemical corrosion.</p>
Fatigue and Creep	<p>Fatigue or creep can lead a part to failure. For example, unanticipated fatigue can result from repeated mechanical or thermal loading.</p>

Beams: Introduction

Euler-Bernoulli Beam Equation

The out-of-plane displacement w of a beam is governed by the Euler-Bernoulli Beam Equation,

$$\frac{d^2}{dx^2} \left[EI \frac{d^2 w}{dx^2} \right] = p$$

where p is the distributed loading (force per unit length) acting in the same direction as y (and w), E is the Young's modulus of the beam, and I is the area moment of inertia of the beam's cross section.

If E and I do not vary with x along the length of the beam, then the beam equation simplifies to,

$$EI \frac{d^4 w}{dx^4} = p$$

Origin of the Beam Equation

The Euler beam equation arises from a combination of four distinct subsets of beam theory: the [kinematic](#), [constitutive](#), [force resultant](#), and [equilibrium](#) definition equations.

The outcome of each of these segments is summarized here:

Kinematics :	$\chi = -\theta = -\frac{dw}{dx}$
Constitutive :	$\sigma(x, y) = E \cdot \varepsilon(x, y)$
Resultants :	$M(x) = \iint y \cdot \sigma(x, y) \cdot dy \cdot dz$ $V(x) = \iint \sigma_{xy}(x, y) \cdot dy \cdot dz$
Equilibrium :	$\frac{dM}{dx} = V \quad \frac{dV}{dx} = -p$

To relate the beam's out-of-plane displacement w to its pressure loading p , we combine the results of the four beam sub-categories in the order shown,

Kinematics \rightarrow Constitutive \rightarrow Resultants \rightarrow Equilibrium = Beam Equation

We'll demonstrate this hierarchy by working backwards. We first combine the 2 equilibrium equations to eliminate V ,

$$\frac{d^2 M}{dx^2} = -p$$

Next replace the moment resultant M with its definition in terms of the direct stress σ ,

$$\frac{d^2}{dx^2} \left(\iint y \cdot \sigma \cdot dy \cdot dz \right) = -p$$

Use the constitutive relation to eliminate σ in favor of the strain ε , and then use kinematics to replace ε in favor of the normal displacement w ,

$$\frac{d^2}{dx^2} \left(E \iint y \cdot \varepsilon \cdot dy \cdot dz \right) = -p \quad \frac{d^2}{dx^2} \left(E \frac{d\chi}{dx} \iint y^2 \cdot dy \cdot dz \right) = -p$$

$$\frac{d^2}{dx^2} \left(E \frac{d^2 w}{dx^2} \iint y^2 \cdot dy \cdot dz \right) = p$$

As a final step, recognizing that the integral over y^2 is the definition of the beam's area moment of inertia I ,

$$I = \iint y^2 \cdot dy \cdot dz$$

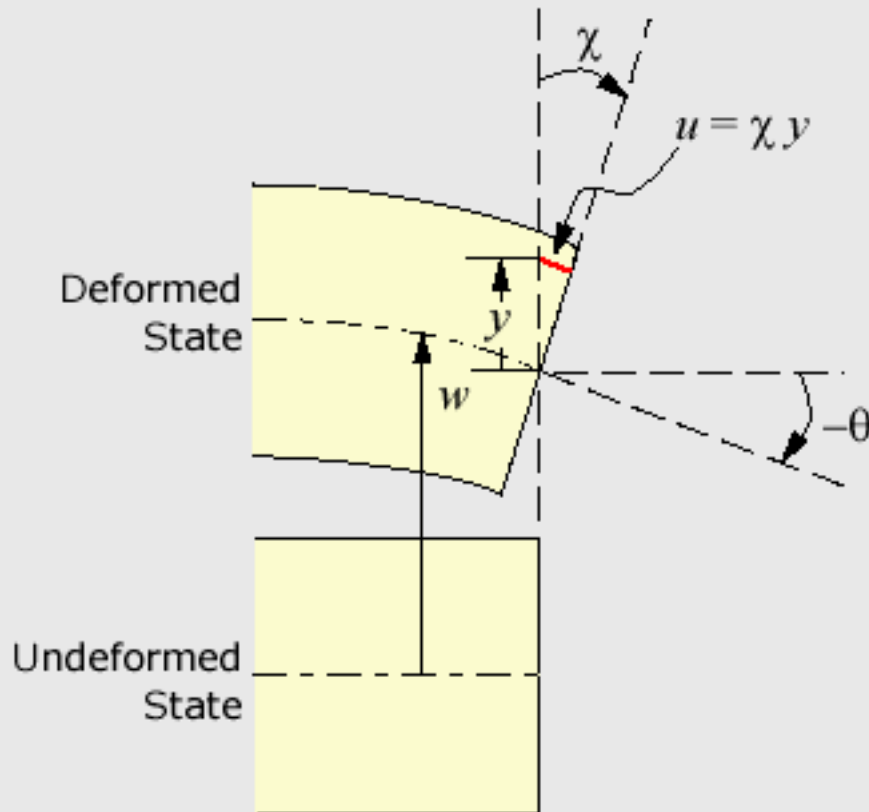
allows us to arrive at the Euler-Bernoulli beam equation,

$$\frac{d^2}{dx^2} \left[EI \frac{d^2 w}{dx^2} \right] = p$$

Beams: Kinematics

Kinematics

Kinematics describe how the beam's deflections are tracked. We've already mentioned the out-of-plane displacement w , the distance the beam's neutral plane moves from its resting (unloaded) position. Out-of-plane displacement is usually accompanied by a rotation of the beam's neutral plane, defined as θ , and by a rotation of the beam's cross section, χ ,



What we really need to know is the displacement in the x -direction across a beam cross section, $u(x,y)$, from which we can find the direct strain $\varepsilon(x,y)$ by the equation,

$$\varepsilon = \frac{du}{dx}$$

To do so requires that we make a few assumptions on just how a beam cross section rotates. For the Euler beam, the assumptions were given by Kirchoff and dictate how the "normals" behave (*normals* are lines perpendicular to the beam's neutral plane and are thus embedded in the beam's cross sections).

Kirchoff Assumptions

1. Normals remain straight (they do not bend)
2. Normals remain unstretched (they keep the same length)
3. Normals remain normal (they always make a right angle to the neutral plane)

With the normals straight and unstretched, we can safely assume that there is negligible strain in the y direction. Along with normals remaining normal to the neutral plane, we can make the x and y dependence in $u(x,y)$ explicit via a simple geometric expression,

$$u(x, y) = \chi(x) \cdot y$$

With explicit x dependence in u , we can find the direct strain throughout the beam,

$$\varepsilon(x, y) = \frac{d\chi}{dx} \cdot y$$

Finally, again with normals always normal, we can tie the cross section rotation χ to the neutral plane rotation θ , and eventually to the beam's displacement w ,

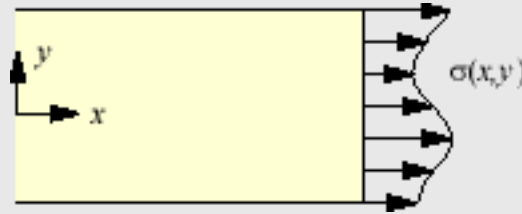
$$\chi = -\theta = -\frac{dw}{dx}$$

Beams: Constitutive Equation

Constitutive

The constitutive equation describes how the direct stress σ and direct strain ε within the beam are related.

Direct means perpendicular to a beam cross section; if we were to cut the beam at a given location, we would find a distribution of direct stress acting on the beam face,



Beam theory typically uses the simple 1-dimensional Hooke's Equation,

$$\sigma(x, y) = E \cdot \varepsilon(x, y)$$

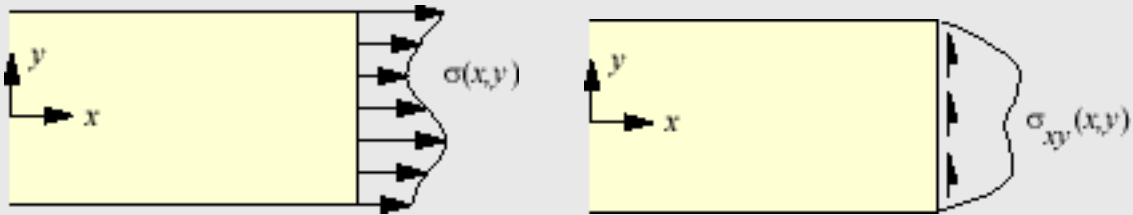
Note that the stress and strain are functions of the entire beam cross section (i.e. they can vary with y).

Beams: Force Resultants

Resultants

Force resultants are a convenient means for tracking the *important* stresses in a beam. They are analogous to the moments and forces of statics theory, in that their influence is felt throughout the beam (as opposed to just a local effect). Their convenience lies in them being only functions of x , whereas stresses in the beam are functions of x and y .

If we were to cut a beam at a point x , we would find a distribution of direct stresses $\sigma(y)$ and shear stresses $\sigma_{xy}(y)$,



Each little portion of direct stress acting on the cross section creates a moment about the neutral plane ($y = 0$). Summing these individual moments over the area of the cross-section is the definition of the moment resultant M ,

$$M(x) = \iint y \cdot \sigma(x, y) \cdot dy \cdot dz$$

where z is the coordinate pointing in the direction of the beam width (out of the screen). Summing the shear stresses on the cross-section is the definition of the shear resultant V ,

$$V(x) = \iint \sigma_{xy}(x, y) \cdot dy \cdot dz$$

There is one more force resultant that we can define for completeness. The sum of all direct stresses acting on the cross-section is known as N ,

$$N(x) = \iint \sigma(x, y) \cdot dy \cdot dz$$

$N(x)$ is the total direct force within the beam at some point x , yet it does not play a role in (linear) beam theory since it does not cause a displacement w . Instead, it plays a role in the axial displacement of rods and bars.

By inverting the definitions of the force resultants, we can find the direct stress distribution in the beam due to bending,

$$\sigma(x, y) = \frac{My}{I}$$

Note that the bending stress in beam theory is linear through the beam thickness. The maximum bending stress occurs at the point furthest away from the neutral axis, $y = c$,

$$\sigma_{max} = \frac{Mc}{I}$$

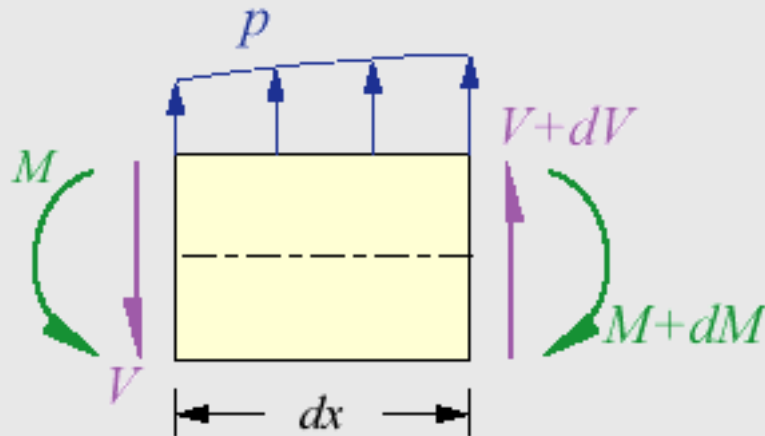
What about the other non-linear direct stresses shown acting on the beam cross section? The average value of the direct stress is contained in N and does not contribute to beam theory. The remaining stresses (after the average and linear parts are subtracted away) are *self-equilibrating* stresses. By a somewhat circular argument, they are self-equilibrating precisely because they do not contribute to M or N , and therefore they do not play a global role. On the contrary, self-equilibrating loads are confined to have only a localized effect as mandated by **Saint-Venant's Principle**.

Beams: Force Equilibrium

Force Equilibrium

The equilibrium equations describe how the beam carries external pressure loads with its internal stresses. Rather than deal with these stresses themselves, we choose to work with the resultants since they are functions of x only (and not of y).

To enforce equilibrium, consider the balance of forces and moments acting on a small section of beam,



Equilibrium in the y -direction gives the equation for the shear resultant V ,

$$\frac{dV}{dx} = -p$$

Moment equilibrium about a point on the right side of the beam gives the equation for the moment resultant M ,

$$\frac{dM}{dx} = V$$

Note that the pressure load p does not contribute to the moment equilibrium equation.

Beams: Symbols

Notation and symbols commonly used in [Euler-Bernoulli beam theory](#) are summarized below:

Independent Parameters

Quantity	Symbol	Object	Units
elastic modulus	E	scalar	N/m ²
area moment of inertia	I	scalar	m ⁴
<i>line pressure</i> load	p	scalar	N/m
concentrated force load	P	scalar	N
location along beam	x	scalar	m
height above neutral surface	y	scalar	m

Dependent Parameters

Quantity	Symbol	Object	Units
transverse displacement	w	scalar	m
slope of neutral surface	θ	scalar	radian
rotation of beam cross section	χ	scalar	radian
moment resultant	M	scalar	N-m
shear resultant	V	scalar	N
stress	σ	scalar	N/m ²
strain	ϵ	scalar	1

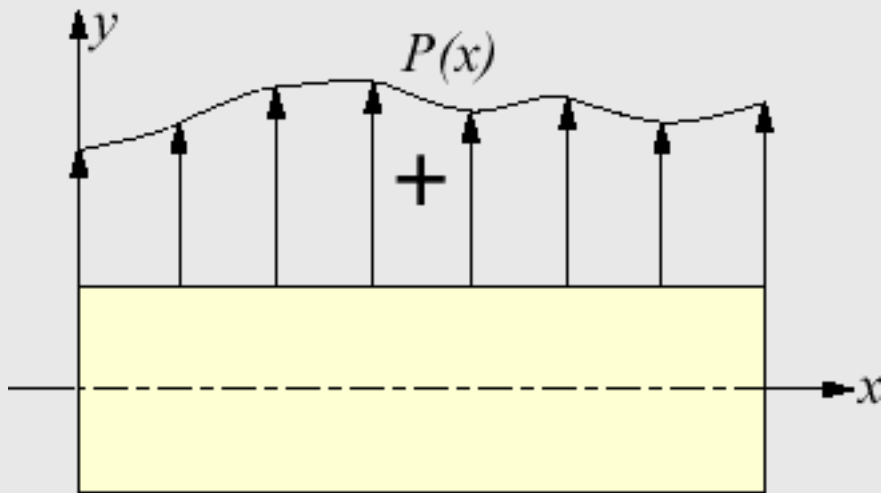
Neutral Plane - Plane in the beam where all bending stresses are zero. It is the reference point for the integral defining the Area Moment of Inertia, I .

Line Pressure Load - Out of convenience, beam theory defines the units for a pressure load as [N/m], as opposed to real pressure units [N/m²]. To convert a real pressure to this line pressure for beams, multiply the true pressure by the beam width,

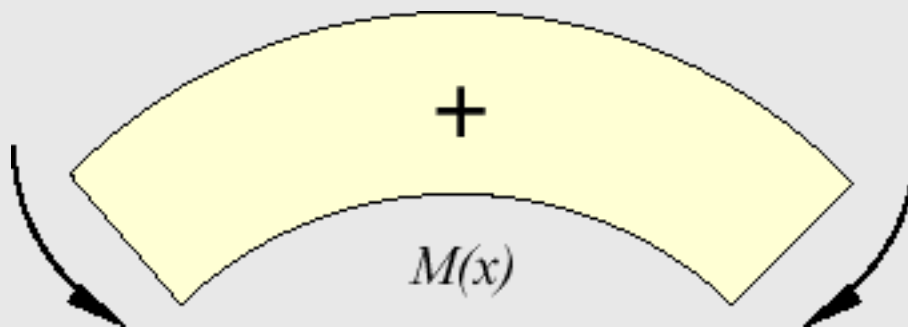
$$P_{beam} = P_{true} \cdot b$$

Beams: Sign Convention

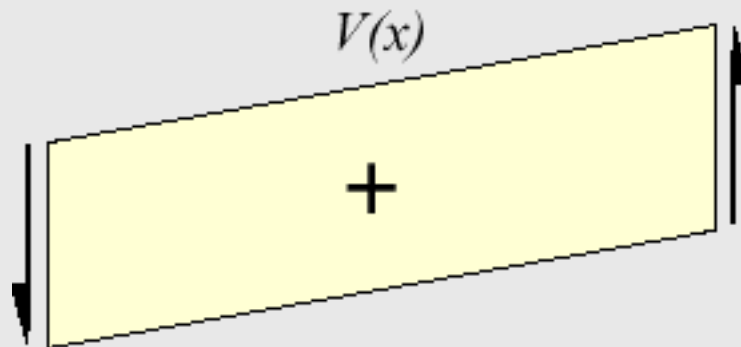
The sign convention used in the beam section is defined in the following illustrations. Notice that unlike some publications on this subject, the loading term $P(x)$ points to the same direction as y -axis. Please refer to the [Euler-Bernoulli beam theory](#) for more details.



The loading P and deflection y directions indicated are positive



The bending moment M direction indicated is positive



The shear force V direction indicated is positive.

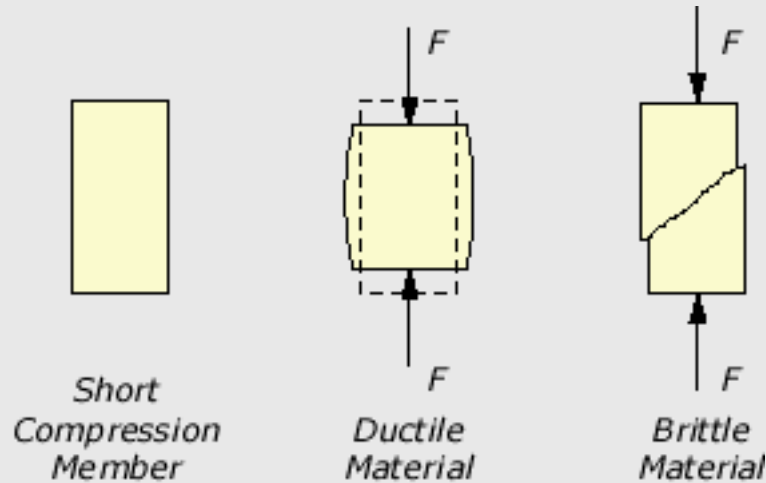
Buckling: Introduction

Compression Members

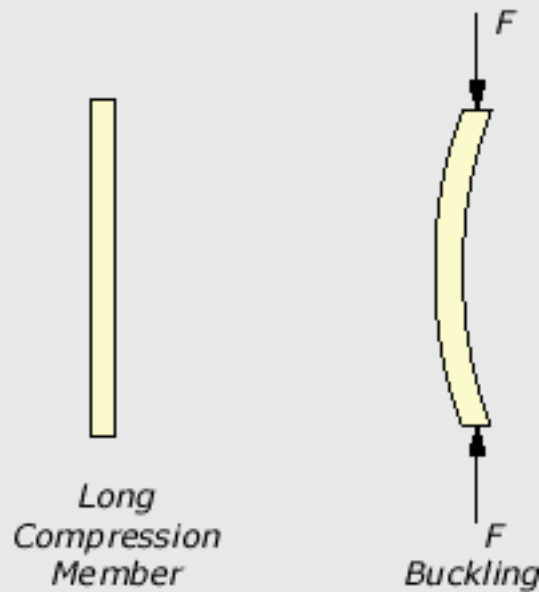
Compression members, such as columns, are mainly subjected to axial forces. The principal stress in a compression member is therefore the normal stress,

$$\sigma = \frac{F}{A}$$

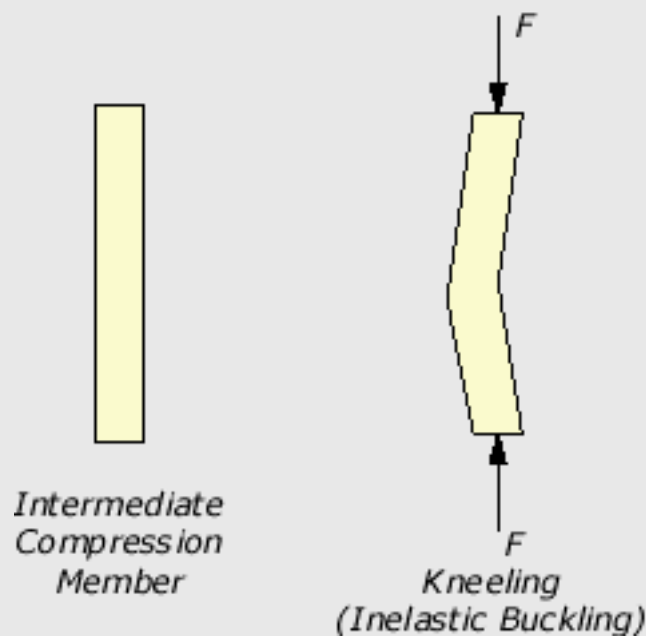
The failure of a *short compression member* resulting from the compression axial force looks like,



However, when a compression member becomes longer, the role of the geometry and stiffness ([Young's modulus](#)) becomes more and more important. For a *long (slender) column*, [buckling](#) occurs way before the normal stress reaches the strength of the column material. For example, pushing on the ends of a business card or bookmark can easily reproduce the buckling.



For an *intermediate length compression member*, **kneeling** occurs when some areas yield before buckling, as shown in the figure below.

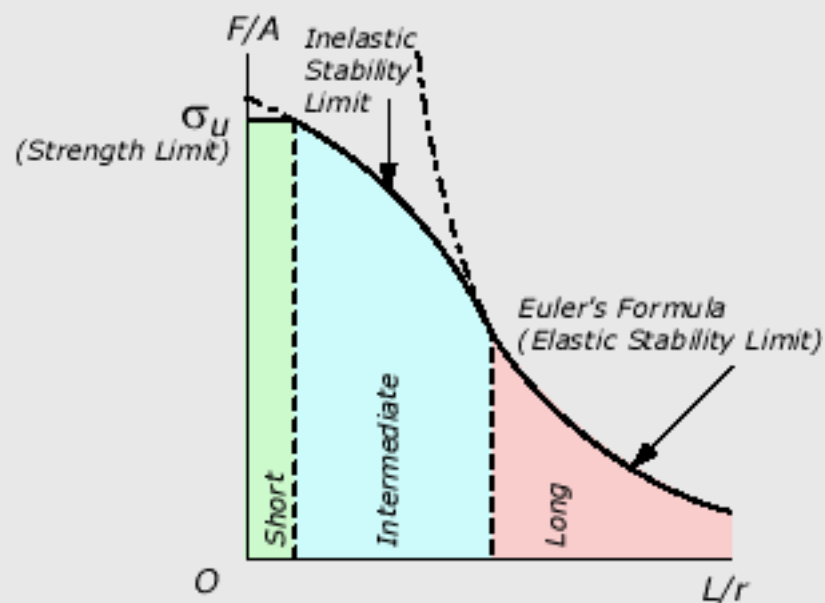


In summary, the failure of a compression member has to do with the strength and stiffness of the material and the geometry (slenderness ratio) of the member. Whether a compression member is considered short, intermediate, or long depends on these factors. More quantitative discussion on these factors can be found in the next section.

Design Considerations

In practice, for a given material, the allowable stress in a compression member depends on the slenderness ratio L_{eff}/r and can be divided into three regions: short, intermediate, and long.

Short columns are dominated by the strength limit of the material. Intermediate columns are bounded by the [inelastic limit](#) of the member. Finally, long columns are bounded by the elastic limit (i.e. [Euler's formula](#)). These three regions are depicted on the stress/slenderness graph below,



The short/intermediate/long classification of columns depends on both the geometry (slenderness ratio) and the material properties (Young's modulus and yield strength). Some common materials used for columns are listed below:

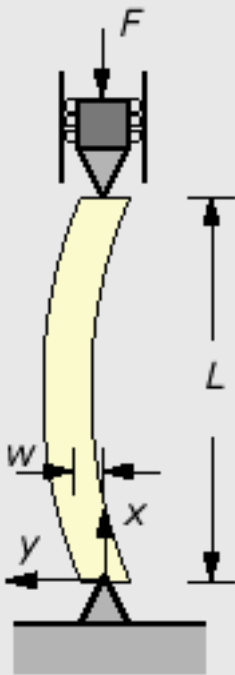
Material	Short Column (Strength Limit)	Intermediate Column (Inelastic Stability Limit)	Long Column (Elastic Stability Limit)
	Slenderness Ratio ($SR = L_{eff} / r$)		
Structural Steel	$SR < 40$	$40 < SR < 150$	$SR > 150$
Aluminum Alloy AA 6061 - T6	$SR < 9.5$	$9.5 < SR < 66$	$SR > 66$
Aluminum Alloy AA 2014 - T6	$SR < 12$	$12 < SR < 55$	$SR > 55$
Wood	$SR < 11$	$11 < SR < (18 \sim 30)$	$(18 \sim 30) < SR < 50$

In the table, L_{eff} is the [effective length](#) of the column, and r is the radius of gyration of the

cross-sectional area, defined as $r = \sqrt{\frac{I}{A}}$.

Buckling: Critical Load

Euler's Formula



Consider a long **simply-supported** column under an external axial load F , as shown in the figure to the left. The **critical buckling load** ([elastic stability limit](#)) is given by Euler's formula,

$$F_{cy} = \frac{EI\pi^2}{L^2}$$

where E is the Young's modulus of the column material, I is the area moment of inertia of the cross-section, and L is the length of the column.

Note that the critical buckling load decreases with the square of the column length.

Simply supported column subjected to axial load F

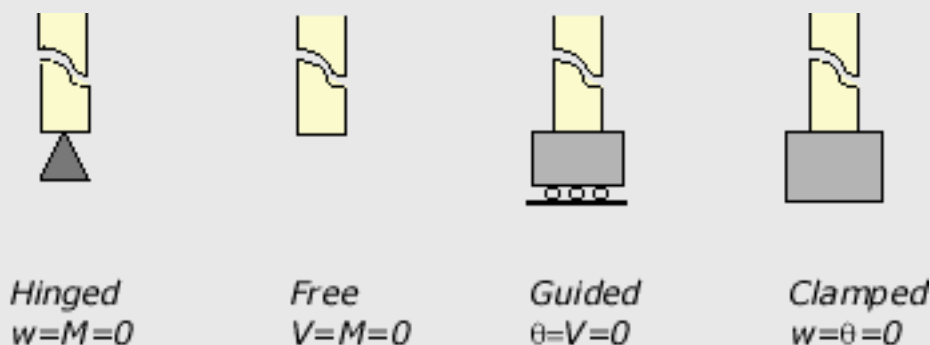
Extended Euler's Formula

In general, columns do not always terminate with simply-supported ends. Therefore, the formula for the critical buckling load must be generalized.

The generalized equation takes the form of Euler's formula,

$$F_{cy} = \frac{EI\pi^2}{L_{\text{eff}}^2}$$

where the effective length of the column L_{eff} depends on the boundary conditions. Some common boundary conditions are shown in the schematics below:



The following table lists the effective lengths for columns terminating with a variety of boundary condition combinations. Also listed is a mathematical representation of the buckled mode shape.

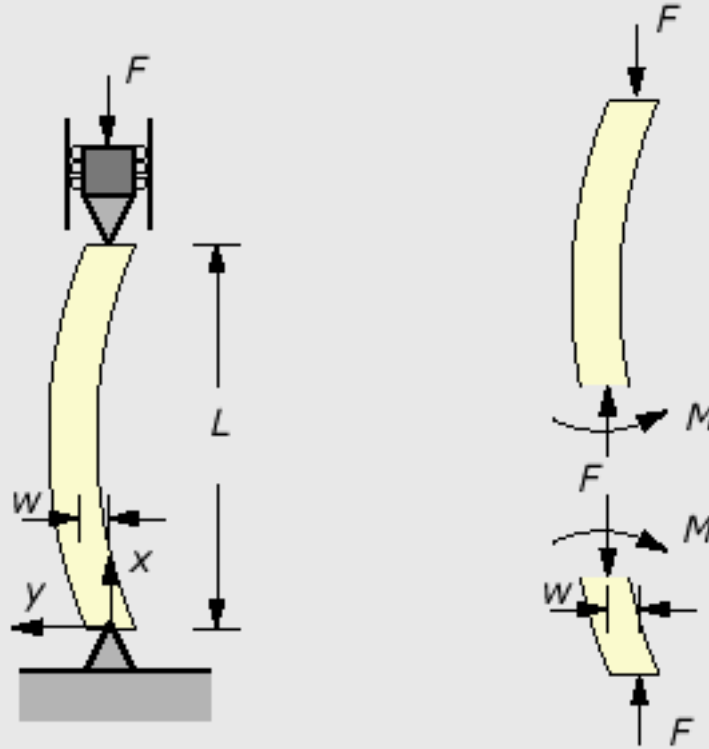
Boundary Conditions	Theoretical Effective Length L_{eff}^T	Engineering Effective Length L_{eff}^E	Buckling Mode Shape
Free-Free	L	$(1.2 \cdot L)$	$\sin \frac{\pi x}{L}$
Hinged-Free	L	$(1.2 \cdot L)$	$\sin \frac{\pi x}{L}$
Hinged-Hinged (Simply-Supported)	L	L	$\sin \frac{\pi x}{L}$
Guided-Free	$2 \cdot L$	$(2.1 \cdot L)$	$\sin \frac{\pi x}{2L}$
Guided-Hinged	$2 \cdot L$	$2 \cdot L$	$\cos \frac{\pi x}{2L}$
Guided-Guided	L	$1.2 \cdot L$	$\cos \frac{\pi x}{L}$
Clamped-Free (Cantilever)	$2 \cdot L$	$2.1 \cdot L$	$1 - \cos \frac{\pi x}{2L}$
Clamped-Hinged	$0.7 \cdot L$	$0.8 \cdot L$	$\sin kx - kL \cos kx + kL \left(1 - \frac{x}{L}\right)$ where $k = 1.4318 \frac{\pi}{L}$
Clamped-Guided	L	$1.2 \cdot L$	$1 - \cos \frac{\pi x}{L}$
Clamped-Clamped	$0.5 \cdot L$	$0.65 \cdot L$	$1 - \cos \frac{2\pi x}{L}$

In the table, L represents the actual length of the column. The effective length is often used in column design by design engineers.

Buckling: Elastic Buckling

Governing Equation for Elastic Buckling

Consider a buckled simply-supported column of length L under an external axial compression force F , as shown in the left schematic below. The transverse displacement of the buckled column is represented by w .



*Simply supported column
subjected to axial load F*

Free body diagram

The right schematic shows the forces and moments acting on a cross-section in the buckled column. Moment equilibrium on the lower free body yields a solution for the internal bending moment M ,

$$Fw - M = 0$$

Recall the relationship between the moment M and the transverse displacement w for an [Euler-Bernoulli](#) beam,

$$\begin{aligned} M &= EI \frac{d\chi}{dx} = -EI \frac{d\theta}{dx} \\ &= -EI \frac{d^2w}{dx^2} \end{aligned}$$

Eliminating M from the above two equations results in the governing equation for the buckled slender column,

$$\frac{d^2w}{dx^2} + \frac{F}{EI}w = 0$$

Buckling Solutions

The governing equation is a [second order homogeneous ordinary differential equation with constant coefficients](#) and can be solved by the method of characteristic equations. The solution is found to be,

$$w(x) = A \sin mx + B \cos mx$$

where $m^2 = \frac{F}{EI}$. The coefficients A and B can be determined by the two boundary conditions

$$w(0) = w(L) = 0, \text{ which yields,}$$

$$\begin{cases} B = 0 \\ A \sin mL = 0 \end{cases}$$

The coefficient B is always zero, and for most values of m^*L the coefficient A is required to be zero. However, for special cases of m^*L , A can be nonzero and the column can be buckled. The restriction on m^*L is also a restriction on the values for the loading F ; these special values are mathematically called eigenvalues. All other values of F lead to trivial solutions (i.e. zero deformation).

$$\sin mL = 0$$

$$\Rightarrow m = n \frac{\pi}{L} \text{ where } n = 1, 2, 3, \dots$$

$$\Rightarrow F = EI \left(\frac{n\pi}{L} \right)^2$$

The lowest load that causes buckling is called critical load ($n = 1$).

$$F_{cy} = \frac{EI\pi^2}{L^2}$$

The above equation is usually called **Euler's formula**. Although Leonard Euler did publish the governing *equation* in 1744, J. L. Lagrange is considered the first to show that a non-trivial solution exists only when n is an *integer*. Thomas Young then suggested the critical load ($n = 1$) and pointed out the solution was valid when the column is *slender* in his 1807 book. The "slender" column idea was not quantitatively developed until A. Considère performed a series of 32 tests in 1889.

The shape function for the buckled shape $w(x)$ is mathematically called an eigenfunction, and is given by,

$$w(x) = A \sin \left(\frac{n\pi x}{L} \right)$$

Recall that this eigenfunction is strictly valid only for simply-supported columns.

- Note:**
1. Boundary conditions other than simply-supported will result in [different critical loads and mode shapes](#).
 2. The buckling mode shape is valid only for small deflections, where the material is still within its elastic limit.
 3. The critical load will cause buckling for slender, long columns. In contrast, failure will occur in short columns when the strength of material is exceeded. Between the long and short column limits, there is a region where buckling occurs after the stress exceeds the proportional limit but is still below the ultimate strength. These columns are classified as *intermediate* and their failure is called [inelastic buckling](#).
 4. Whether a column is short, intermediate, or long depends on its geometry as well as the stiffness and strength of its material. This concept is addressed in the columns [introduction](#) page.

Buckling: Inelastic Buckling

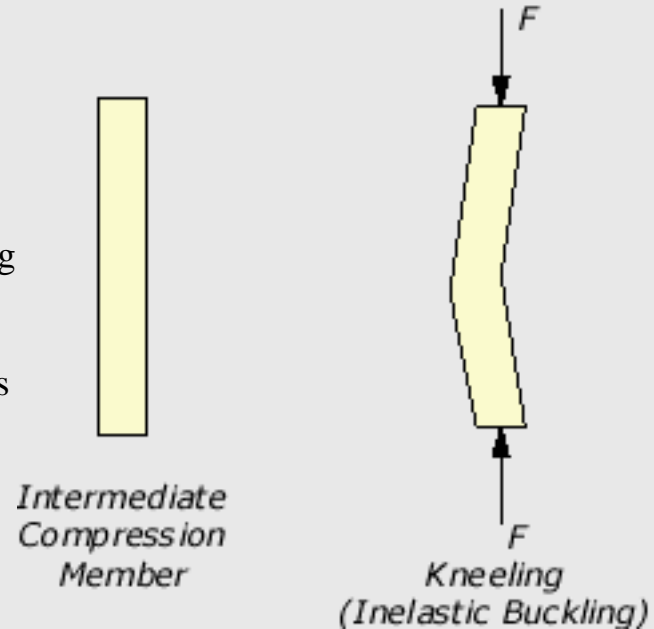
Intermediate Columns

The strength of a compression member (column) depends on its geometry (slenderness ratio L_{eff}/r) and its material properties (stiffness and strength).

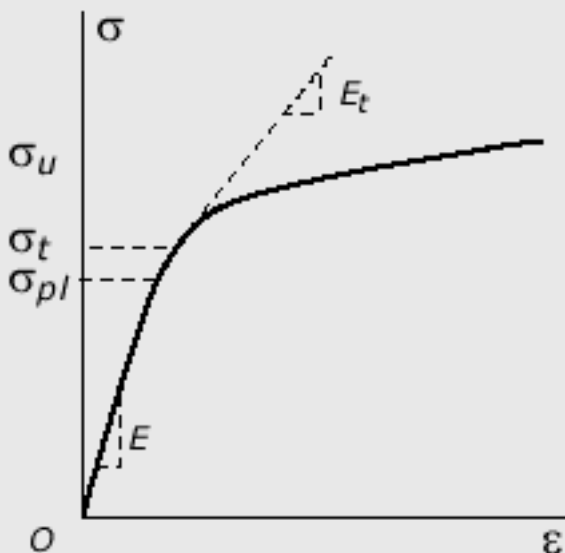
The [Euler formula](#) describes the critical load for [elastic buckling](#) and is valid only for *long columns*. The ultimate compression strength of the column material is not geometry-related and is valid only for *short columns*.

In between, for a column with *intermediate* length, buckling occurs after the stress in the column exceeds the proportional limit of the column material and before the stress reaches the ultimate strength. This kind of situation is called **inelastic buckling**.

This section discusses some commonly used inelastic buckling theories that fill the gap between short and long columns.



Tangent-Modulus Theory



Suppose that the critical stress σ_t in an intermediate column exceeds the proportional limit of the material σ_{pl} . The Young's modulus at that particular stress-strain point is no longer E . Instead, the Young's modulus decreases to the local tangent value, E_t .

Replacing the Young's modulus E in the [Euler's formula](#) with the tangent modulus E_t , the critical load becomes,

$$F_t = \frac{E_t I \pi^2}{L_{eff}^2}$$

The corresponding critical stress is,

$$\sigma_t = \frac{E_t \pi^2}{(L_{eff}/r)^2}$$

- Note:**
1. The proportional limit σ_{pl} , rather than the yield stress σ_y , is used in the formula. Although these two are often arbitrarily interchangeable, the yield stress is about equal to or slightly larger than the proportional limit for common engineering materials. However, when the forming process is taken into account, the residual stresses caused by processing can not be neglected and the proportional limit may drop up to 50% with respect to the yield stress in some wide-flange sections.
 2. The tangent-modulus theory tends to underestimate the strength of the column, since it uses the tangent modulus once the stress on the concave side exceeds the proportional limit while the convex side is still below the elastic limit.
 3. The tangent-modulus theory oversimplifies the inelastic buckling by using only one tangent modulus. In reality, the tangent modulus depends on the stress, which is a function of the bending moment that varies with the displacement w .

Reduced-Modulus Theory

The Reduced Modulus theory defines a reduced Young's modulus E_r to compensate for the underestimation given by the [tangent-modulus theory](#).

For a column with *rectangular cross section*, the reduced modulus is defined by,

$$E_r = \frac{4EE_t}{(\sqrt{E} + \sqrt{E_t})^2}$$

where E is the value of Young's modulus below the proportional limit. Replacing E in [Euler's formula](#) with the reduced modulus E_r , the critical load becomes,

$$P_y = \frac{E_r I \pi^2}{L_{\text{eff}}^2}$$

The corresponding critical stress is,

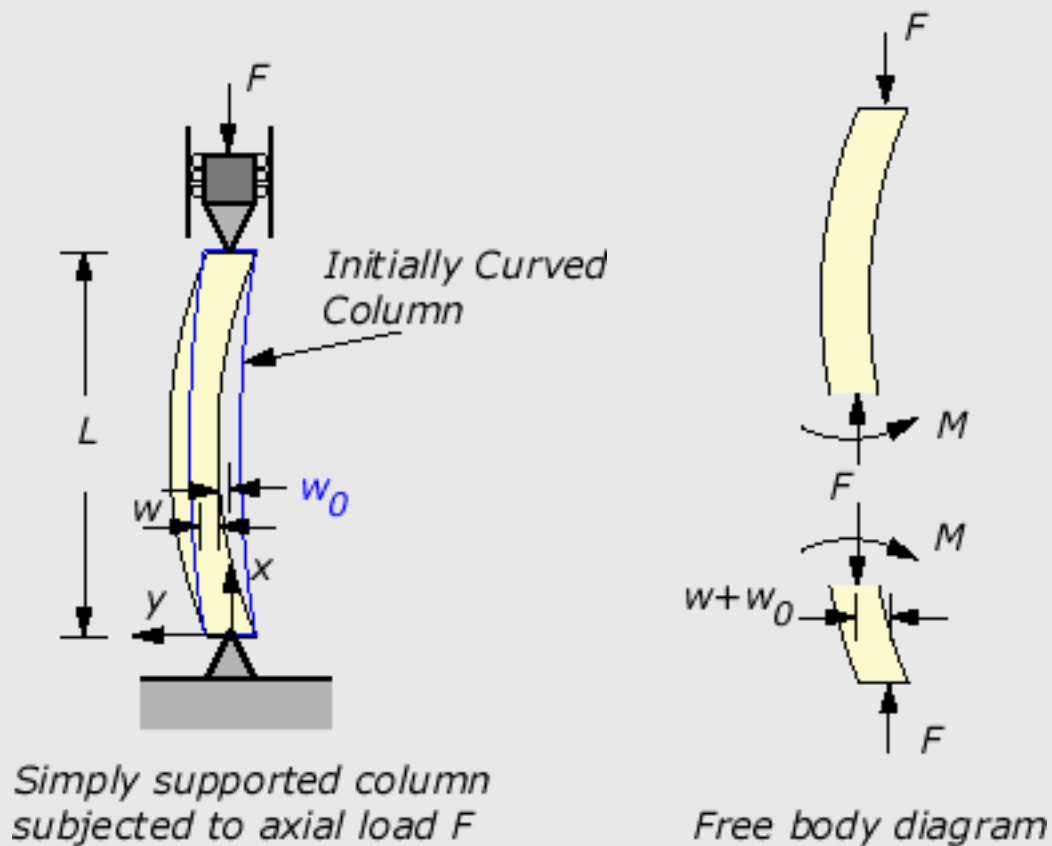
$$\sigma_y = \frac{E_r \pi^2}{(L_{\text{eff}}/r)^2}$$

- Note:**
1. The reduced-modulus theory tends to overestimate the strength of the column, since it is based on stiffness reversal on the convex side of the column.
 2. The reduced-modulus theory oversimplifies the inelastic buckling by using only one tangent modulus. In reality, the tangent modulus depends on the stress which is a function of the bending moment that varies with the displacement w .

Buckling: Initially Curved Columns

Imperfections in Columns

Consider a curved column with an initial shape $w_0(x)$ that is under an axial load F ,



Enforcing the moment equilibrium of the free-body yields the equation,

$$F(w + w_0) - M = 0$$

The governing equation for the transverse displacement w can then be expressed as,

$$\frac{d^2 w}{dx^2} + \frac{F}{EI} w = -\frac{F w_0}{EI}$$

The above equation contains a non-homogeneous term $-F w_0/EI$. To simplify the problem, we assume that the initial curve is a simple half sine wave with an amplitude a ,

$$w_0 = a \sin \frac{\pi x}{L}$$

The general solution is found to be,

$$w = \frac{aF \sin \frac{\pi x}{L}}{EI \left(\frac{\pi}{L} \right)^2 - F} = \frac{F}{F_{cy} - F} \left(a \sin \frac{\pi x}{L} \right)$$

$$= \frac{F/F_{cy}}{1 - F/F_{cy}} w_0$$

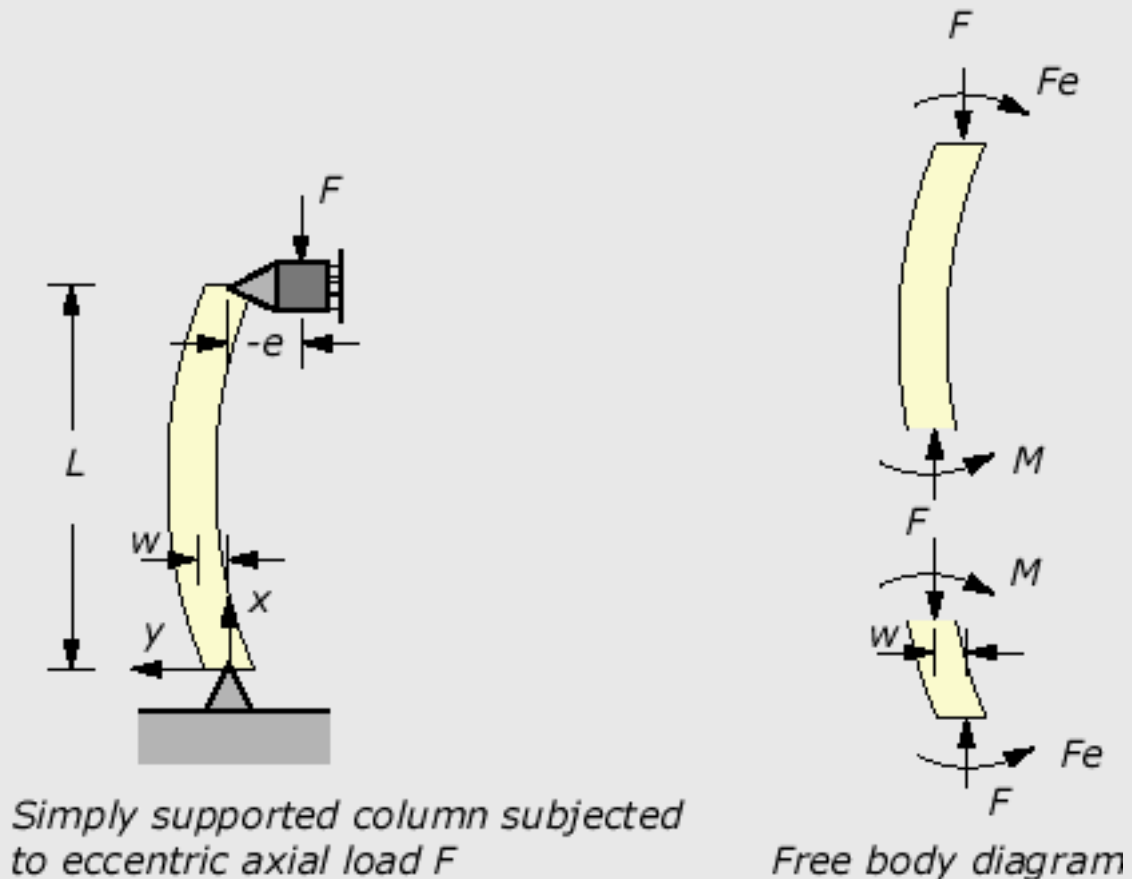
where $F_{cy} = \frac{EI\pi^2}{L^2}$ is the [critical load](#) of a straight column.

Buckling: Eccentric Loads

Eccentric Axial Load

In the case of an ideal column under an axial load, the column remains straight until the critical load is reached. However, the load is not always applied at the centroid of the cross section, as is assumed in Euler buckling [theory](#). This section analyzes a simply-supported column under an eccentric axial load.

Consider a column of length L subject to an axial force F . On one end of the column, the force F is applied a distance e from the central column axis, as shown in the schematic below.



Simply supported column subjected to eccentric axial load F

Free body diagram

Balance the moments on the free-body diagram on the right requires that,

$$Fw + Fe - M = 0$$

The governing equation for the column's transverse displacement w can then be written as,

$$\frac{d^2 w}{dx^2} + \frac{F}{EI} w = -\frac{Fe}{EI}$$

where M was eliminated using [Euler-Bernoulli](#) beam theory. The above equation contains a non-homogeneous term $-Fe/EI$ and its general solution is,

$$w(x) = A \sin mx + B \cos mx - e$$

where $m^2 = \frac{F}{EI}$. The coefficients A and B depend on the boundary conditions. For a simply supported column the boundary conditions are,

$$w(0) = w(L) = 0$$

The solution for the column's displacement is therefore,

$$w = e \left(\tan \frac{mL}{2} \sin mx + \cos mx - 1 \right)$$

where $m^2 = \frac{F}{EI}$.

Secant Formula

In practice, engineers are usually interested in the maximum stress rather than the displacement curve alone. The secant formula discussed in this section derives the maximum stress from the displacement formula obtained in the previous section.

The normal stress in the column results from both the direct axial load F and the bending moment M resulting from the eccentricity e of the force application,

$$\begin{aligned} \sigma &= \frac{F}{A} + \frac{My}{I} \\ &= \frac{F}{A} + \frac{F(w+e)y}{I} \end{aligned}$$

where A is the cross-section area, and I is the moment of inertia of the cross section.

The maximum stress is located at the extreme fiber on the concave side ($y = c$) of the middle point ($x = L/2$) of the column,

$$\begin{aligned} \sigma_{\max} &= \frac{F}{A} + \frac{M_{\max} y_{\max}}{I} \\ &= \frac{F}{A} + \frac{F(w_{\max} + e)c}{I} \end{aligned}$$

where,

$$w_{\max} = w\left(\frac{L}{2}\right) = e \left(\sec \frac{mL}{2} - 1 \right)$$

(obtained by applying basic trigonometric relations to the displacement formula in the previous

section). The parameter c is the distance from the centroidal axis to the extreme fiber on the concave side of the column.

Expanding the formula for the maximum stress, we have,

$$\begin{aligned}\sigma_{\max} &= \frac{F}{A} + \frac{F \cdot e \cdot c}{I} \sec \frac{mL}{2} \\ &= \frac{F}{A} \left[1 + \frac{A}{I} ec \sec \left(\sqrt{\frac{F}{EI}} \frac{L}{2} \right) \right]\end{aligned}$$

The radius of gyration r is defined as $r = \sqrt{\frac{I}{A}}$. Working r into the above stress equation results in the **secant formula** for maximum stress,

$$\sigma_{\max} = \frac{F}{A} \left(1 + \frac{ec}{r^2} \sec \left(\sqrt{\frac{F}{EA}} \frac{L}{2r} \right) \right)$$

The secant formula indicates that in addition to the axial load F and cross-section area A , the maximum stress also depends on the eccentricity ratio ec/r^2 and the slenderness ratio L/r .

Note: 1. The secant formula can be used to compute the allowable normal stress for a given design,

$$\sigma_{\max} = \frac{F}{A} \left(1 + \frac{ec}{r^2} \sec \left(\sqrt{\frac{F}{EA}} \frac{L}{2r} \right) \right) \leq \sigma_{\text{allow}}$$

where σ_{allow} is the maximum allowable stress (e.g. yield stress).

2. If the eccentricity e is zero, the secant formula no longer applies. In this case [Euler's formula](#) must be used for slender beams.

$$\text{Min} \{ F/A, F_{\text{cr}}/A \} \leq \sigma_{\text{allow}}$$

$$\begin{aligned}\text{where } \frac{F_{\text{cr}}}{A} &= \sigma_{\text{cr}} = \frac{EI\pi^2}{AL^2} \\ &= \frac{E\pi^2}{(L/r)^2}\end{aligned}$$

Buckling: Beam Columns

Beam-Column Equation

The out-of-plane transverse displacement w of a beam subject to in-plane loads is governed by the equation,

$$\frac{d^2}{dx^2} \left[EI \frac{d^2 w}{dx^2} \right] + \frac{d}{dx} \left[f \frac{dw}{dx} \right] = p$$

where p is a distributed transverse load (force per unit length) acting in the positive- y direction, f is an axial compression force, E is the Young's modulus of the beam, and I is the area moment of inertia of the beam's cross section. The above equation is sometimes referred to as the **beam-column equation**, since it exhibits behaviors of both beams and columns.

If E and I do not vary with x across the length of the beam and f remains constant, denoted as F , then the beam-column equation can be simplified to,

$$EI \frac{d^4 w}{dx^4} + F \frac{d^2 w}{dx^2} = p$$

Origin of the Beam-Column Equation

Similar to the [Euler-Bernoulli beam equation](#), the beam-column equation arises from four distinct subsets of beam-column theory: kinematics, constitutive, force resultants, and equilibrium.

The outcome of each of these segments is summarized as follows:

<u>Kinematics:</u>	$\chi = -\theta = -\frac{dw}{dx}$
<u>Constitutive:</u>	$\sigma(x, y) = E \cdot \varepsilon(x, y)$
<u>Resultants:</u>	$N(x) = \int \int \sigma_x(x, y) \cdot dy \cdot dz$ $M(x) = \int \int y \cdot \sigma_x(x, y) \cdot dy \cdot dz$ $V(x) = \int \int \sigma_{xy}(x, y) \cdot dy \cdot dz$

Equilibrium:	$\frac{dM}{dx} - N \cdot \chi = V$	$\frac{dV}{dx} = -p$
---------------------	------------------------------------	----------------------

In the equilibrium equations, N is the axial force resulting acting in a tensile manner (opposite in direction to the compressive resultant f).

To relate the beam's out-of-plane displacement w to its pressure loading p , we combine the results of the four sub-categories in the following order:

Kinematics \Rightarrow Constitutive \Rightarrow Resultants \Rightarrow Equilibrium \Rightarrow Beam-Column Equation

This hierarchy can be demonstrated by working backwards. First combine the two equilibrium equations to eliminate V :

$$\frac{d^2 M}{dx^2} - \frac{d}{dx}[N \cdot \chi] = -p$$

Next replace the moment resultant M with its definition in terms of the direct stress σ :

$$\frac{d^2}{dx^2} \left[\int \int y \cdot \sigma_x(x, y) \cdot dy \cdot dz \right] - \frac{d}{dx}[N \cdot \chi] = -p$$

Use the constitutive relation to eliminate σ in favor of the strain ε , and then use kinematics to replace ε in favor of the normal displacement w :

$$\frac{d^2}{dx^2} \left[E \cdot \int \int y \cdot \varepsilon_x \cdot dy \cdot dz \right] - \frac{d}{dx}[N \cdot \chi] = -p$$

$$\text{Since } \varepsilon_x = \frac{du}{dx} = \frac{d}{dx}(\chi \cdot y) = \frac{d\chi}{dx} \cdot y = -\frac{d^2 w}{dx^2} \cdot y$$

$$\Rightarrow \frac{d^2}{dx^2} \left[E \cdot \frac{d\chi}{dx} \cdot \int \int y^2 \cdot dy \cdot dz \right] - \frac{d}{dx}[N \cdot \chi] = -p$$

$$\Rightarrow \frac{d^2}{dx^2} \left[E \cdot \int \int y^2 \cdot dy \cdot dz \cdot \frac{d^2 w}{dx^2} \right] - \frac{d}{dx} \left[N \cdot \frac{dw}{dx} \right] = p$$

As a final step, recognizing that the integral over y^2 is the definition of the beam's area moment of inertia I ,

$$I = \int \int y^2 \cdot dy \cdot dz$$

We arrive at the beam-column equation based on the Euler-Bernoulli beam theory,

$$\frac{d^2}{dx^2} \left[EI \frac{d^2 w}{dx^2} \right] - \frac{d}{dx} \left[N \frac{dw}{dx} \right] = p$$

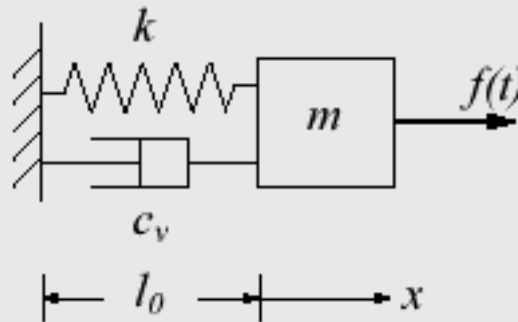
Since columns are usually used as compression members, engineers may be more familiar with the axial *compression* resultant f than the tensile resultant N . Let $f = -N$. The beam-column equation expressed with f is therefore,

$$\frac{d^2}{dx^2} \left[EI \frac{d^2 w}{dx^2} \right] + \frac{d}{dx} \left[f \frac{dw}{dx} \right] = p$$

Dynamics: Introduction

Definition

The simplest vibratory system can be described by a single mass connected to a spring (and possibly a dashpot). The mass is allowed to travel only along the spring elongation direction. Such systems are called *Single Degree-of-Freedom* (SDOF) systems and are shown in the following figure,



Equation of Motion for SDOF Systems

SDOF vibration can be analyzed by Newton's second law of motion, $F = m \cdot a$. The analysis can be easily visualized with the aid of a [free body diagram](#),

$$F = ma$$

The free body diagram shows the mass m with forces kx and $c_v \dot{x}$ acting to the left, and $f(t)$ acting to the right. The resulting acceleration is $m\ddot{x}$.

$$-kx - c_v \dot{x} + f(t) = m\ddot{x}$$

The resulting equation of motion is a [second order, non-homogeneous](#), ordinary differential equation:

$$m\ddot{x} + c_v \dot{x} + kx = f(t)$$

with the initial conditions,

$$\begin{cases} x(t=0) = x_0 \\ \dot{x}(t=0) = v_0 \end{cases}$$

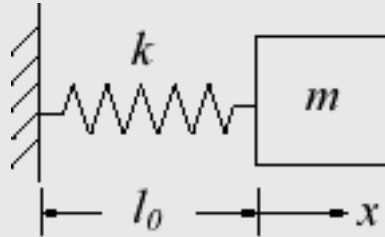
The solution to the general SDOF equation of motion is shown in the [damped SDOF](#) discussion.

Dynamics: Undamped SDOF System

Definition of an Undamped SDOF System

If there is no external force applied on the system, $f(t) = 0$, the system will experience *free vibration*. Motion of the system will be established by an initial disturbance (i.e. initial conditions).

Furthermore, if there is no resistance or damping in the system, $c_v = 0$, the oscillatory motion will continue forever with a constant amplitude. Such a system is termed undamped and is shown in the following figure,



Time Solution for Undamped SDOF Systems

The [equation of motion](#) derived on the [introductory page](#) can be simplified to,

$$m \ddot{x} + kx = 0$$

with the initial conditions,

$$\begin{cases} x(t=0) = x_0 \\ \dot{x}(t=0) = v_0 \end{cases}$$

This equation of motion is a [second order](#), [homogeneous](#), ordinary differential equation (ODE). If the mass and spring stiffness are constants, the ODE becomes a linear homogeneous ODE with constant coefficients and can be solved by the Characteristic Equation method. The characteristic equation for this problem is,

$$ms^2 + k = 0$$

which determines the 2 independent roots for the undamped vibration problem. The final solution (that contains the 2 independent roots from the characteristic equation and satisfies the initial conditions) is,

$$\begin{aligned}
 x(t) &= c_1 e^{i\omega_n t} + c_2 e^{-i\omega_n t} \\
 &= d_1 \cos \omega_n t + d_2 \sin \omega_n t \\
 \Rightarrow x(t) &= x_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t
 \end{aligned}$$

The natural frequency ω_n is defined by,

$$\omega_n = \sqrt{\frac{k}{m}}$$

and depends only on the system mass and the spring stiffness (i.e. any damping will not change the natural frequency of a system).

Alternatively, the solution may be expressed by the equivalent form,

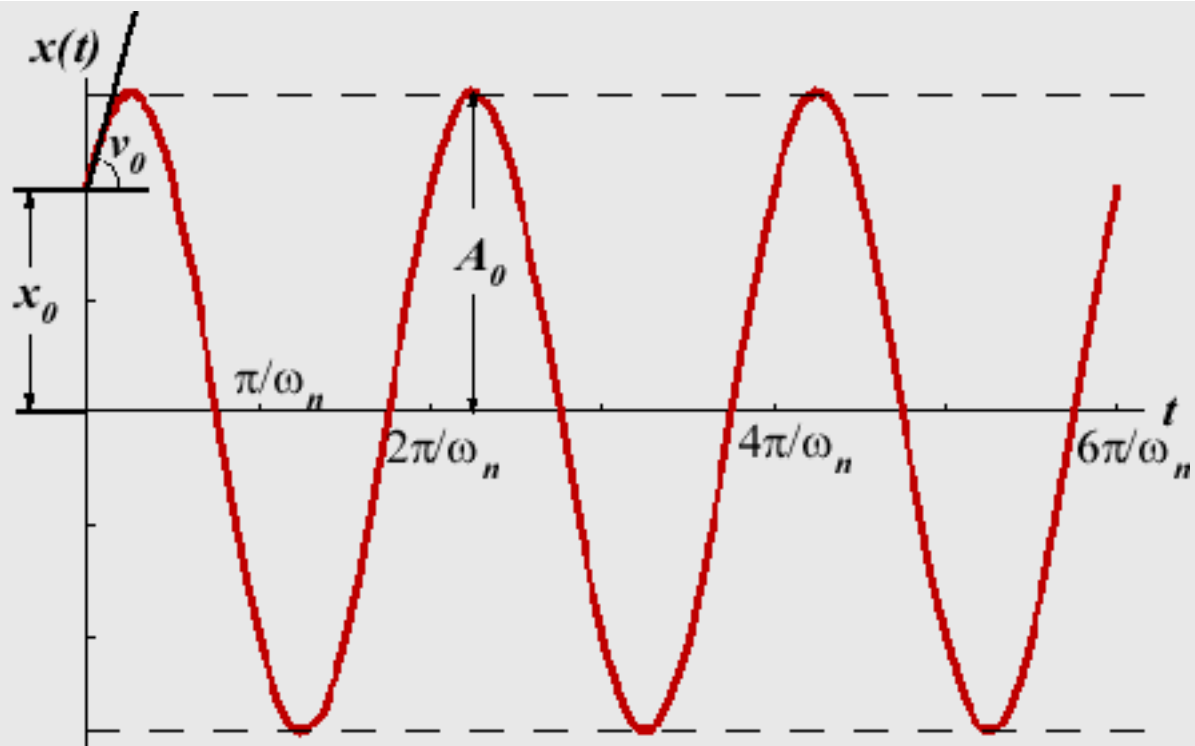
$$x(t) = A_0 \cos(\omega_n t - \phi_0)$$

where the amplitude A_0 and initial phase ϕ_0 are given by,

$$\begin{aligned}
 A_0 &= \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} \\
 \phi_0 &= \tan^{-1}\left(\frac{v_0}{x_0 \omega_n}\right)
 \end{aligned}$$

Sample Time Behavior

The displacement plot of an undamped system would appear as,

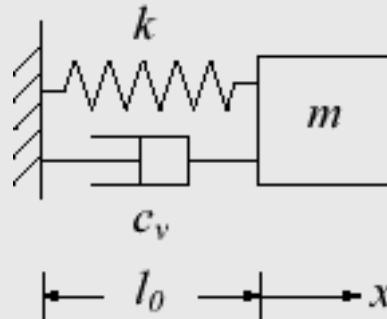


Please note that an assumption of zero damping is typically not accurate. In reality, there almost always exists some resistance in vibratory systems. This resistance will damp the vibration and dissipate energy; the oscillatory motion caused by the initial disturbance will eventually be reduced to zero.

Dynamics: Damped SDOF System

Definition

Free vibration (no external force) of a single degree-of-freedom system with viscous damping can be illustrated as,



Damping that produces a damping force proportional to the mass's velocity is commonly referred to as "viscous damping", and is denoted graphically by a dashpot.

Time Solution for Damped SDOF Systems

For an unforced damped SDOF system, the [general equation of motion](#) becomes,

$$m\ddot{x} + c_v\dot{x} + kx = 0$$

with the initial conditions,

$$\begin{cases} x(t=0) = x_0 \\ \dot{x}(t=0) = v_0 \end{cases}$$

This equation of motion is a [second order](#), [homogeneous](#), ordinary differential equation (ODE). If all parameters (mass, spring stiffness, and viscous damping) are constants, the ODE becomes a linear ODE with constant coefficients and can be solved by the Characteristic Equation method. The characteristic equation for this problem is,

$$ms^2 + c_v s + k = 0$$

which determines the 2 independent roots for the damped vibration problem. The roots to the characteristic equation fall into one of the following 3 cases:

1. If $c_v^2 - 4mk < 0$, the system is termed [underdamped](#). The roots of the characteristic equation are complex conjugates, corresponding to *oscillatory motion* with an *exponential decay* in amplitude.
2. If $c_v^2 - 4mk = 0$, the system is termed [critically-damped](#). The roots of the characteristic equation are repeated, corresponding to *simple decaying motion* with at most *one overshoot* of the system's resting position.
3. If $c_v^2 - 4mk > 0$, the system is termed [overdamped](#). The roots of the characteristic equation are purely real and distinct, corresponding to simple *exponentially decaying motion*.

To simplify the solutions coming up, we define the critical damping c_c , the damping ratio ζ , and the damped vibration frequency ω_d as,

$$c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

$$\zeta = \frac{c_v}{c_c}$$

$$\omega_d = \sqrt{1 - \zeta^2} \omega_n$$

where the natural frequency of the system ω_n is given by,

$$\omega_n = \sqrt{\frac{k}{m}}$$

Note that ω_d will equal ω_n when the damping of the system is zero (i.e. [undamped](#)). The time solutions for the free SDOF system is presented below for each of the three case scenarios.

To obtain the time solution of any free SDOF system (damped or not), use the [SDOF Calculator](#).

Underdamped Systems

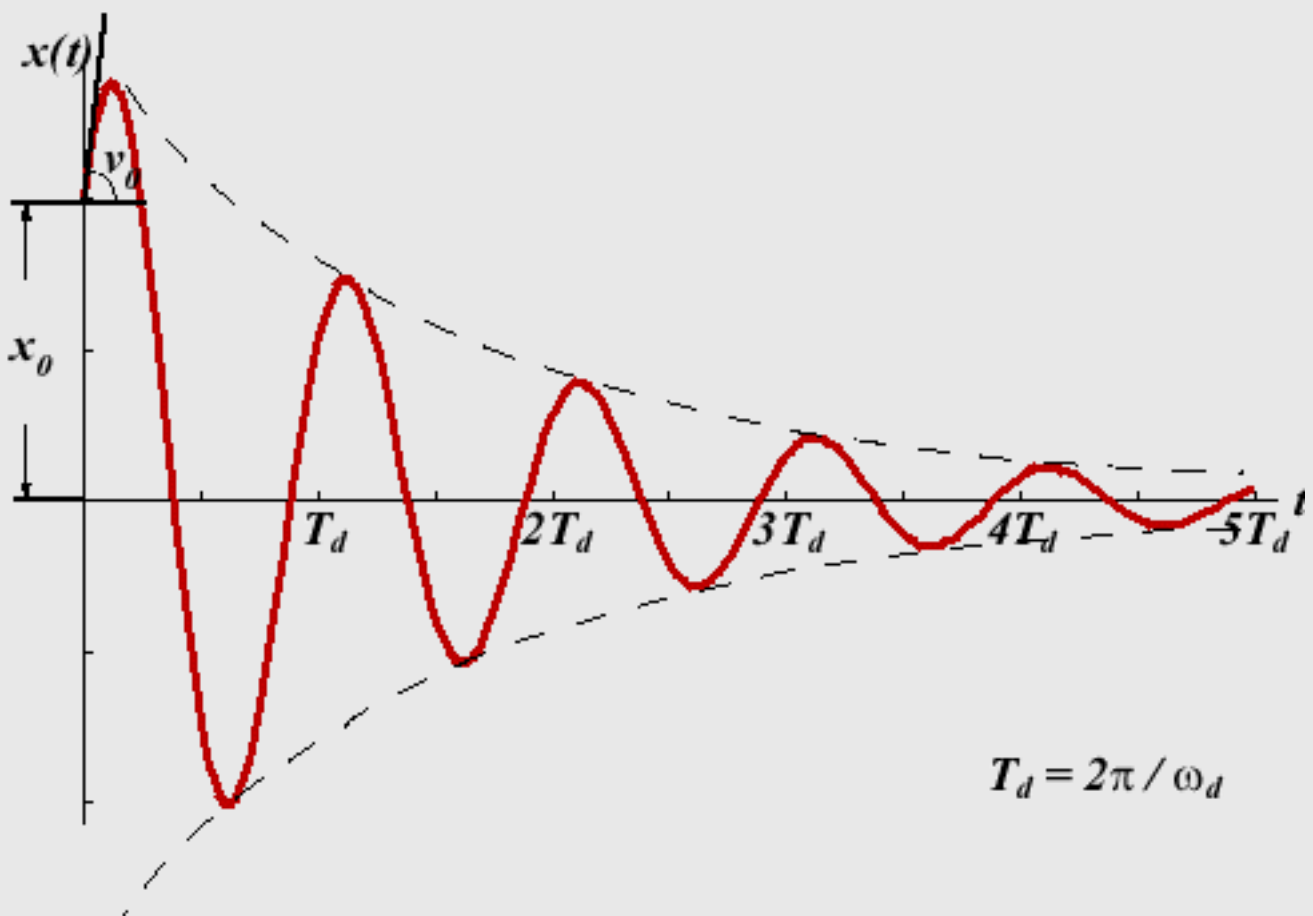
When $c_v^2 - 4mk < 0$ (equivalent to $\zeta < 1$ or $c_v < c_c$), the characteristic equation has a pair of complex conjugate roots. The displacement solution for this kind of system is,

$$\begin{aligned}
 x(t) &= c_1 e^{\left(-\zeta + i\sqrt{1-\zeta^2}\right)\omega_n t} + c_2 e^{\left(-\zeta - i\sqrt{1-\zeta^2}\right)\omega_n t} \\
 &= e^{-\zeta\omega_n t} [d_1 \cos(\omega_d t) + d_2 \sin(\omega_d t)] \\
 \Rightarrow x(t) &= \underbrace{e^{-\zeta\omega_n t}}_{\text{Exponentially decay}} \underbrace{\left[x_0 \cos(\omega_d t) + \frac{v_0 + \zeta\omega_n x_0}{\omega_d} \sin(\omega_d t) \right]}_{\text{Periodic motion}}
 \end{aligned}$$

An alternate but equivalent solution is given by,

$$x(t) = A_0 \underbrace{e^{-\zeta\omega_n t}}_{\text{Exponentially decay}} \underbrace{\cos(\omega_d t - \varphi_0)}_{\text{Periodic}}$$

The displacement plot of an underdamped system would appear as,



Note that the displacement amplitude decays exponentially (i.e. the natural logarithm of the amplitude ratio for any two displacements separated in time by a constant ratio is a constant; long-winded!),

$$\frac{A_k}{A_{k+1}} = \frac{A_0 e^{-\zeta \omega_n (kT_d)} \cos(\varphi_0)}{A_0 e^{-\zeta \omega_n [(k+1)T_d]} \cos(\varphi_0)} = \frac{e^{-\zeta \omega_n (kT_d)}}{e^{-\zeta \omega_n [(k+1)T_d]}} = e^{\zeta \omega_n T_d}$$

$$\Rightarrow \ln \left(\frac{A_k}{A_{k+1}} \right) = \zeta \omega_n T_d = \zeta \omega_n \frac{2\pi}{\omega_d} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

where $T_d = \frac{1}{f_d} = \frac{2\pi}{\omega_d}$ is the period of the damped vibration.

Critically-Damped Systems

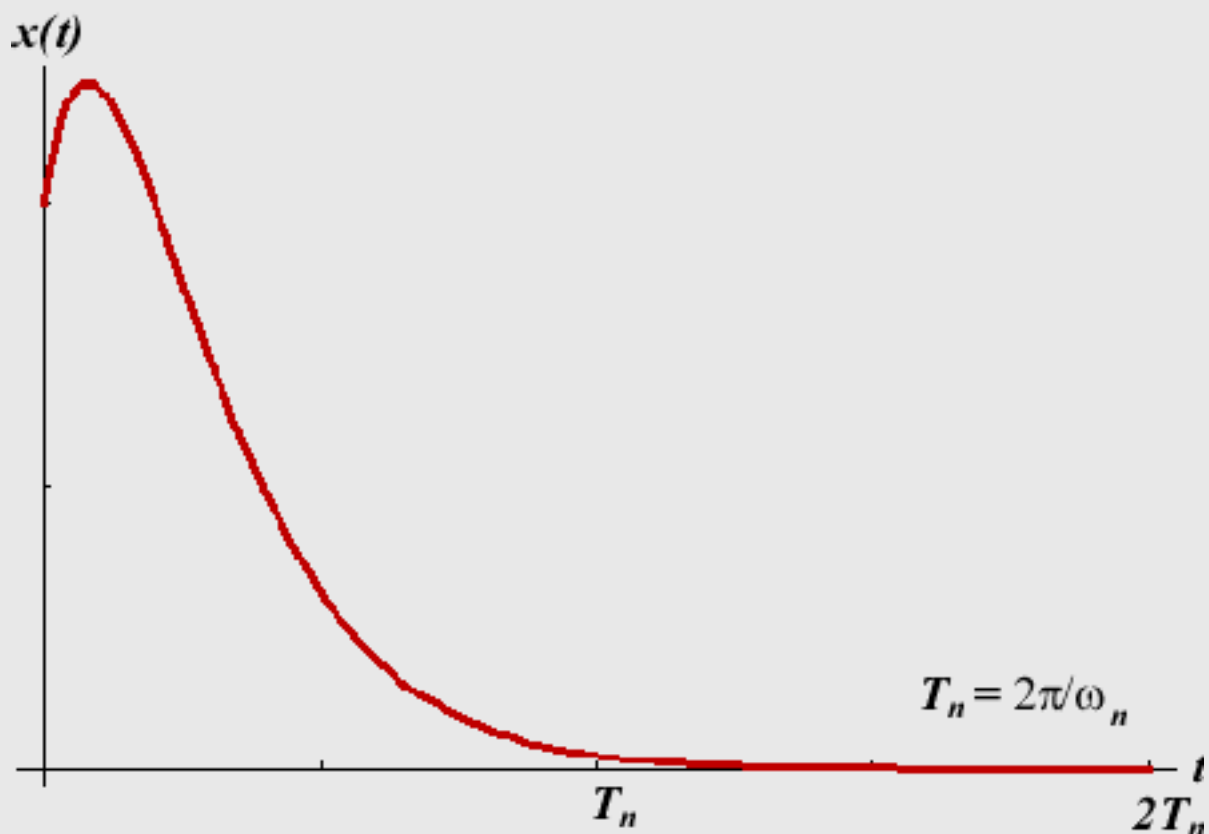
When $c_v^2 - 4mk = 0$ (equivalent to $\zeta = 1$ or $c_v = c_c$), the characteristic equation has repeated real roots. The displacement solution for this kind of system is,

$$x(t) = (c_1 + c_2 t) e^{-\omega_n t}$$

$$\Rightarrow x(t) = e^{-\omega_n t} [x_0 + (v_0 + \omega_n x_0) t]$$

The critical damping factor c_c can be interpreted as the *minimum damping* that results in non-periodic motion (i.e. simple decay).

The displacement plot of a critically-damped system with positive initial displacement and velocity would appear as,



The displacement decays to a negligible level after one natural period, T_n . Note that if the initial velocity v_0 is negative while the initial displacement x_0 is positive, there will exist one overshoot of the resting position in the displacement plot.

Overdamped Systems

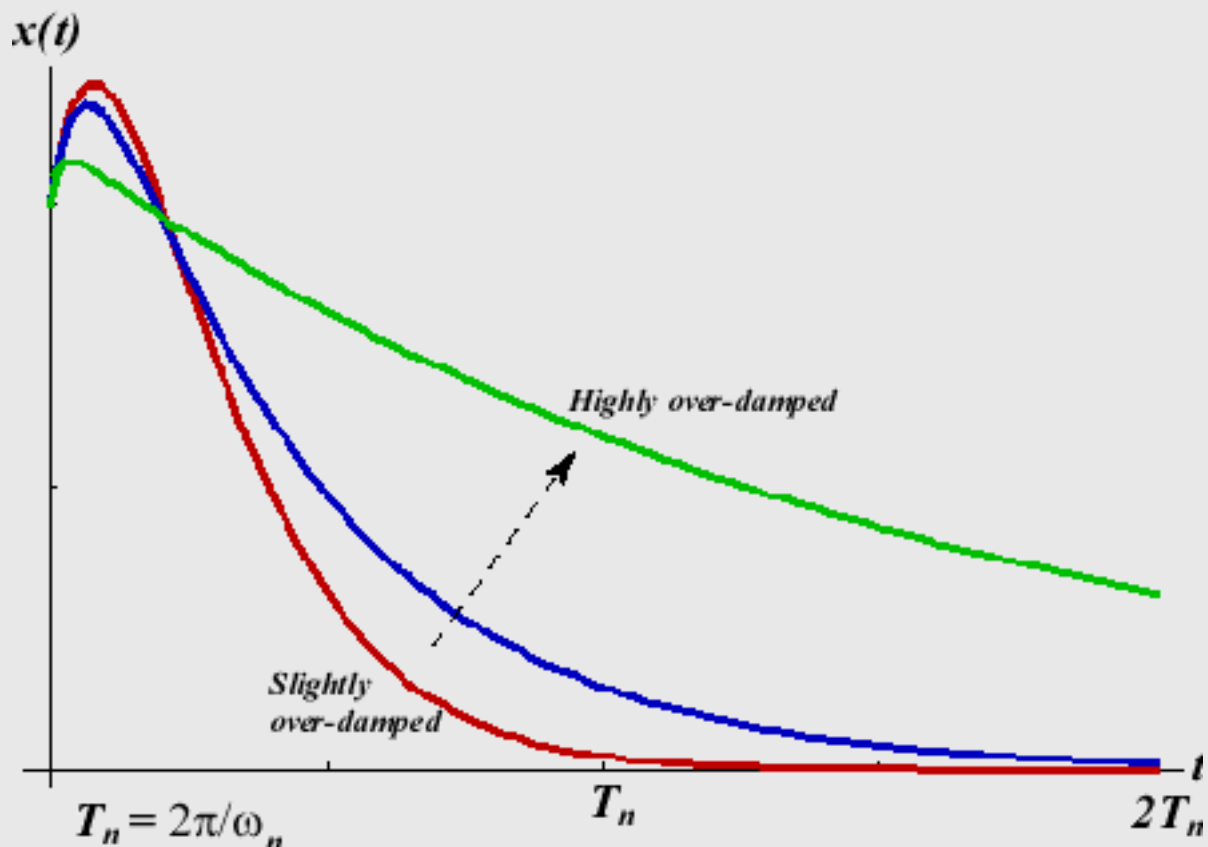
When $c_v^2 - 4mk > 0$ (equivalent to $\zeta > 1$ or $c_v > c_c$), the characteristic equation has two distinct real roots. The displacement solution for this kind of system is,

$$x(t) = c_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right) \omega_n t} + c_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right) \omega_n t}$$

$$\Rightarrow x(t) = \frac{x_0 \omega_n \left(\zeta + \sqrt{\zeta^2 - 1}\right) + v_0 \left(-\zeta + \sqrt{\zeta^2 - 1}\right) \omega_n t}{2 \omega_n \sqrt{\zeta^2 - 1}} e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right) \omega_n t} +$$

$$\frac{-x_0 \omega_n \left(\zeta - \sqrt{\zeta^2 - 1}\right) - v_0 \left(-\zeta - \sqrt{\zeta^2 - 1}\right) \omega_n t}{2 \omega_n \sqrt{\zeta^2 - 1}} e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right) \omega_n t}$$

The displacement plot of an overdamped system would appear as,



The motion of an overdamped system is non-periodic, regardless of the initial conditions. The larger

the damping, the longer the time to decay from an initial disturbance.

If the system is heavily damped, $\zeta \gg 1$, the displacement solution takes the approximate form,

$$x(t) \approx x_0 + \frac{v_0}{2\zeta\omega_n} \left(1 - e^{-2\zeta\omega_n t} \right)$$

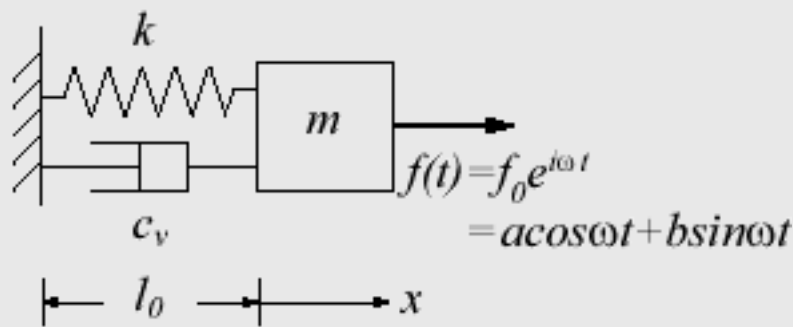
Dynamics: Harmonic Excitation

SDOF Systems under Harmonic Excitation

When a SDOF system is forced by $f(t)$, the solution for the displacement $x(t)$ consists of two parts: the *complimentary solution*, and the *particular solution*. The complimentary solution for the problem is given by the [free vibration discussion](#). The particular solution depends on the nature of the forcing function.

When the forcing function is **harmonic** (i.e. it consists of at most a sine and cosine at the same frequency, a quantity that can be expressed by the complex exponential $e^{i\omega t}$), the method of Undetermined Coefficients can be used to find the particular solution. Non-harmonic forcing functions are handled by [other techniques](#).

Consider the SDOF system forced by the harmonic function $f(t)$,



The particular solution for this problem is found to be,

$$x_p(t) = \frac{f_0}{(k - m\omega^2) + ic\omega} e^{i\omega t}$$

The general solution is given by the sum of the complimentary and particular solutions multiplied by two weighting constants c_1 and c_2 ,

$$x(t) = c_1 \cdot x_c(t) + c_2 \cdot x_p(t)$$

The values of c_1 and c_2 are found by matching $x(t = 0)$ to the initial conditions.

Undamped SDOF Systems under Harmonic Excitation

For an undamped system ($c_v = 0$) the total displacement solution is,

$$x(t) = d_1 \cos \omega_n t + d_2 \sin \omega_n t + \frac{f_0}{k - m\omega^2} e^{i\omega t}$$

$$\Rightarrow x(t) = \left(x_0 - \frac{f_0}{k - m\omega^2} \right) \cos \omega_n t + \left(\frac{v_0 - \frac{i\omega f_0}{k - m\omega^2}}{\omega_n} \right) \sin \omega_n t + \frac{f_0}{k - m\omega^2} e^{i\omega t}$$

If the forcing frequency is close to the natural frequency, $\omega \approx \omega_n$, the system will exhibit **resonance** (very large displacements) due to the near-zeros in the denominators of $x(t)$.

When the forcing frequency is equal to the natural frequency, we cannot use the $x(t)$ given above as it would give divide-by-zero. Instead, we must use L'Hôpital's Rule to derive a solution free of zeros in the denominators,

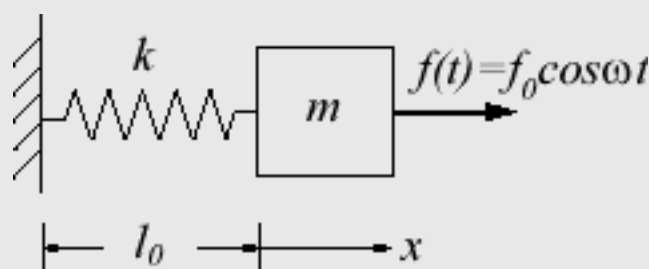
$$x(t) = x_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t +$$

$$\lim_{\omega \rightarrow \omega_n} \left\{ \frac{f_0}{k - m\omega^2} \left(e^{i\omega t} - \cos \omega_n t - i\omega \sin \omega_n t \right) \right\}$$

$$= x_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t - \frac{f_0 \omega_n}{2k} \left(i t e^{i\omega_n t} - i \sin \omega_n t \right)$$

To simplify $x(t)$, let's assume that the driving force consists only of the cosine function,

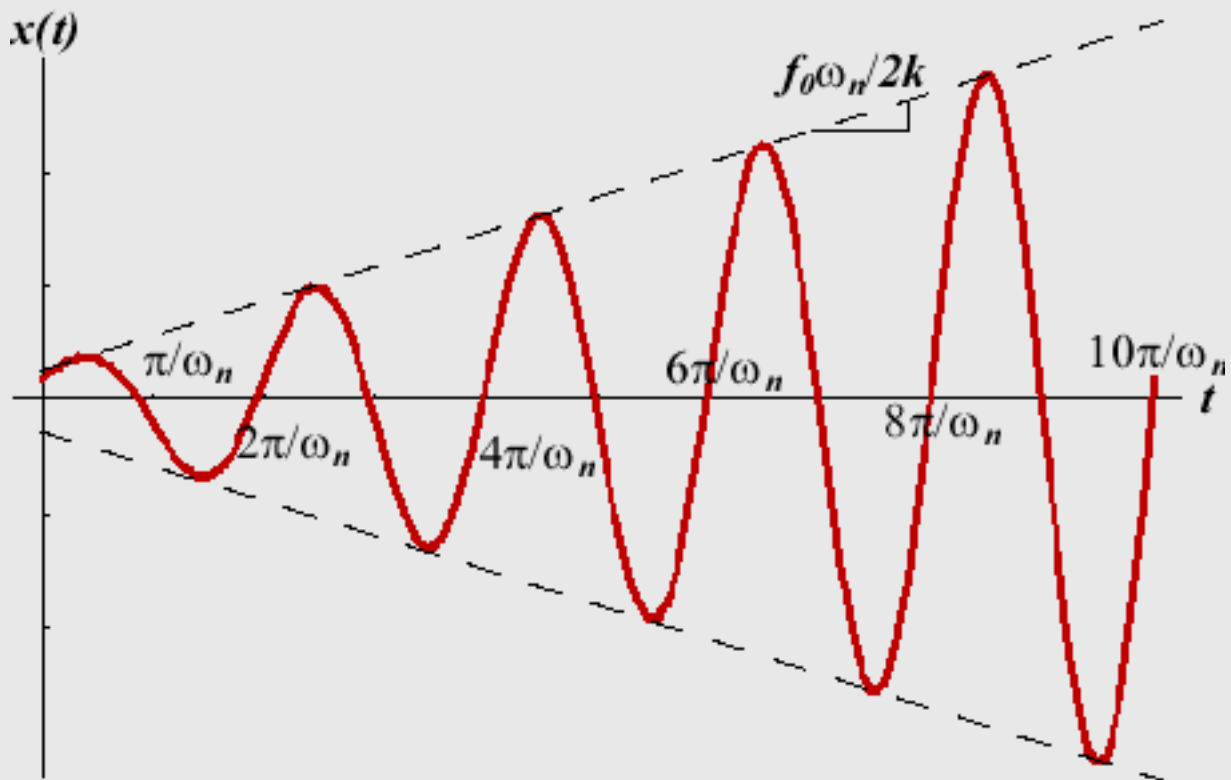
$$f(t) = f_0 \cos \omega t,$$



The displacement solution reduces to,

$$\begin{aligned}
 x(t) &= x_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t + \\
 &\lim_{\omega \rightarrow \omega_n} \left\{ \frac{f_0}{k - m\omega^2} (\cos \omega t - \cos \omega_n t) \right\} \\
 &= \underbrace{x_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t}_{\text{Free vibration (complementary) solution}} + \underbrace{\frac{f_0 \omega_n t}{2k}}_{\text{Amplitude linearly increased}} \sin \omega_n t
 \end{aligned}$$

This solution contains one term multiplied by t . This term will cause the displacement amplitude to increase linearly with time as the forcing function pumps energy into the system, as shown in the following displacement plot,



The maximum displacement of an undamped system forced at its resonant frequency will increase unbounded according to the solution for $x(t)$ above. However, real systems will inject additional physics once displacements become large enough. These additional physics (nonlinear plastic deformation, heat transfer, buckling, etc.) will serve to limit the maximum displacement exhibited by the system, and allow one to escape the "sudden death" impression that such systems will immediately fail.

Dynamics: General Forcing

Common Methods for Solving General Forcing Conditions

The vibration of a linear system under a general forcing function $f(t)$ can be solved by either of the following,

- [Convolution Integral](#)
- [Laplace Transform](#)

These are briefly discussed in the following sections.

Convolution Integral

Consider a *linear* system where (by definition) the response to a general excitation can be obtained by a *superposition* of simple excitation responses.

One of the simplest excitations is the delta function (or *impulse function*) which has the important property:

$$\int_{t_1}^{t_2} f(\tau)\delta(t-\tau)d\tau = \begin{cases} f(t) & t_1 \leq t \leq t_2 \\ 0 & \text{otherwise} \end{cases}$$

This property states that a general forcing function $f(t)$ defined in the interval (t_1, t_2) can be expressed as the superposition (or integration) of many delta functions with magnitude $f(\tau)$ positioned throughout the excitation time interval.

Hence, if we define our forcing function $f(t)$ as equaling the sum of delta functions when inside the time interval t_1 to t_2 ,

$$f(t) = \int_{t_1}^{t_2} f(\tau)\delta(t-\tau)d\tau$$

and equaling zero otherwise, the displacement response $x(t)$ of a linear SDOF system subjected to $f(t)$ is then given by,

$$x(t) = \int_{t_1}^{t_2} f(\tau)g(t-\tau)d\tau$$

The function $g(t)$ is the *impulse response* of the system. By definition, a system's impulse response is equal to $x(t)$ when $f(t)$ is just a single delta function,

$$f(t) = \delta(t)$$

Laplace Transform

When the response of a linear system is difficult to obtain in the time domain (for example, say the Convolution Integral did not permit a closed form solution), the Laplace transform can be used to transform the problem into the frequency domain. The Laplace Transform of $h(t)$ is defined by,

$$\mathcal{L}\{h(t)\} \equiv H(s) = \int_0^{\infty} e^{-st} h(t) dt$$

Transforming a SDOF equation of motion converts an ODE into an algebraic expression which is typically much easier to solve. After obtaining a solution for the displacement $X(s)$ in the frequency domain, the inverse Laplace Transform is used to find $x(t)$, where the inverse transform is defined by,

$$\mathcal{L}^{-1}\{H(s)\} = h(t)$$

Vibration analysis often makes use of the frequency domain method, especially in the field of control theory, since the method is straightforward and systematic. However, the inverse transform can be difficult to find for complex systems.

Dynamics: Glossary

Glossary

Beat Phenomenon

When a two degree-of-freedom system has two closely spaced natural frequencies, ω_{n1} and ω_{n2} , vibration kinetic energy will transfer from one degree-of-freedom to the other in a periodic fashion. The frequency of this transfer is known as the beat frequency, given by $(\omega_{n1} - \omega_{n2}) / 2$.

Critical Damping

The minimum damping that results in *non-periodic motion* of a system under free vibration.

Damping Ratio

The ratio of a system's actual damping to its critical damping. When less than 1, the system is underdamped and will exhibit ringing when disturbed. When larger than 1, the system is overdamped and disturbances will die out without ringing.

Degree-of-Freedom

In the simplest of cases, a degree-of-freedom is an independent displacement or rotation that a system may exhibit. A degree-of-freedom for a system is analogous to an independent variable for a mathematical function. All system degrees-of-freedom must be specified to fully characterize the system at any given time.

Free Body Diagram

A schematic isolating an object (or part of an object) from its environment for the purpose of revealing all external forces and moments acting on the object. Free body diagrams are helpful in applying Newton's 2nd Law of motion to objects.

Maxwell's Reciprocity Theorem

For two identically-sized forces applied at the distinct points A and B on a linear structure, Maxwell's Reciprocity Theorem states that the displacement at A caused by the force at B is the same as the displacement at B caused by the force at A. As a result, the flexibility matrix (and its inverse, the stiffness matrix) of linear systems is symmetric.

Natural Frequency

A frequency where a system resonance exists. If excited at this frequency, the system will exhibit very large displacements (for low damping levels). If the system is undamped, then vibrations can occur at the natural frequency without any external excitation indefinitely.

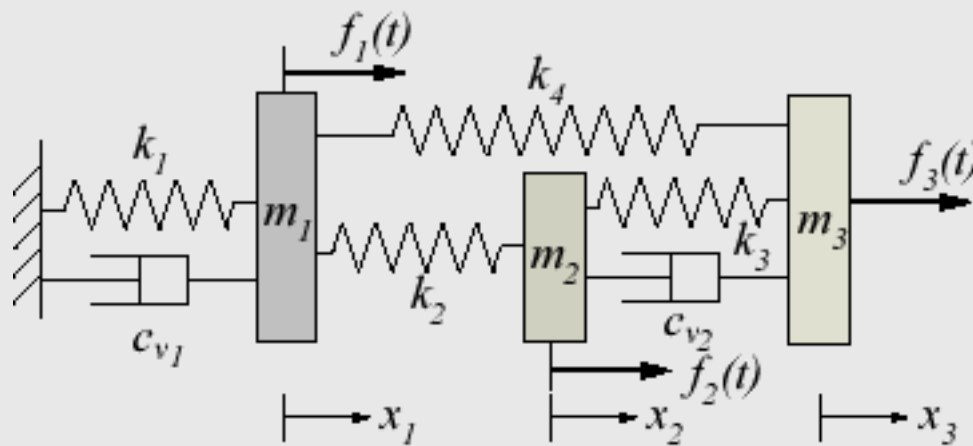
Resonance

A condition where very little energy input into a structure results in a very large displacement (for low damping levels). By definition, resonances occur at the natural frequencies of a system.

Dynamics: Multiple Degree of Freedom

Multiple Degree-of-Freedom Example

Consider the 3 degree-of-freedom system,

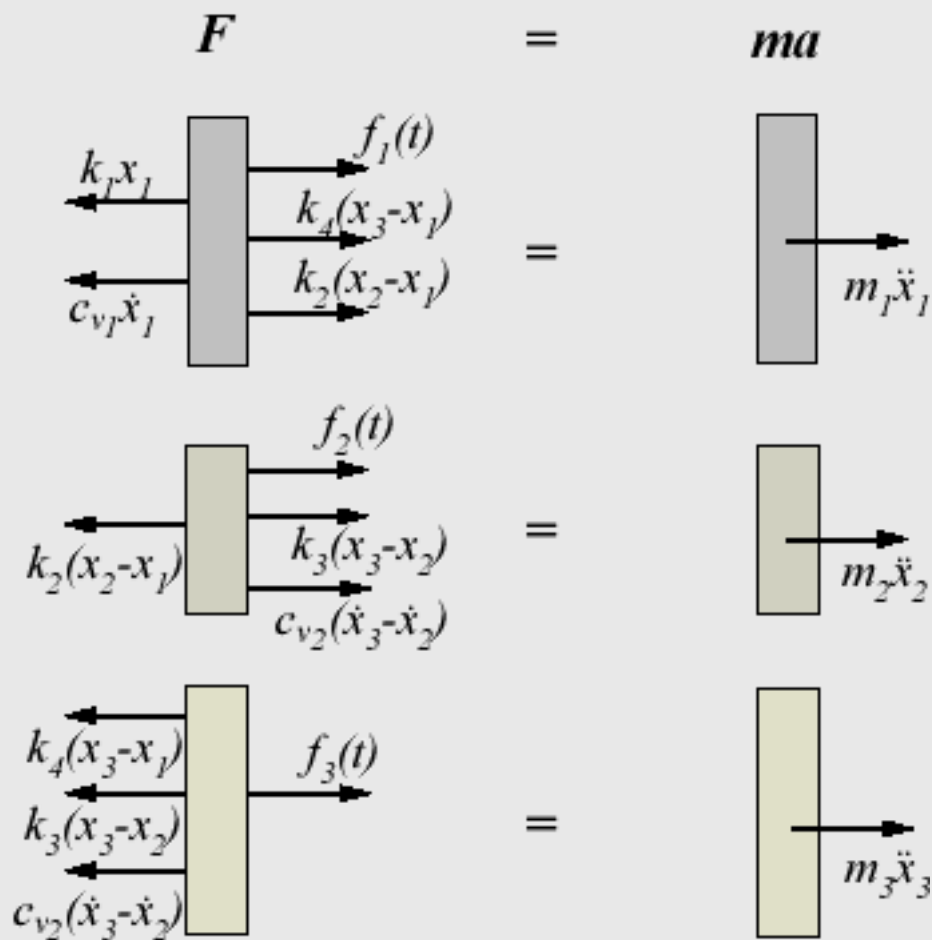


There are 3 [degrees of freedom](#) in this problem since to fully characterize the system we must know the positions of the three masses (x_1 , x_2 , and x_3).

Three [free body diagrams](#) are needed to form the equations of motion. However, it is also possible to [form the coefficient matrices directly](#), since each parameter in a mass-dashpot-spring system has a very distinguishable role.

Equations of Motion from Free Body Diagrams

The equations of motion can be obtained from *free body diagrams*, based on the Newton's second law of motion, $F = m \cdot a$.



The equations of motion can therefore be expressed as,

$$\begin{aligned}
 m_1 \ddot{x}_1 + c_{v1} \dot{x}_1 + (k_1 + k_2 + k_4) x_1 - k_2 x_2 - k_4 x_3 &= f_1(t) \\
 m_2 \ddot{x}_2 + c_{v2} \dot{x}_2 - c_{v2} \dot{x}_3 + (k_2 + k_3) x_2 - k_2 x_1 - k_3 x_3 &= f_2(t) \\
 m_3 \ddot{x}_3 + c_{v2} \dot{x}_3 - c_{v2} \dot{x}_2 + (k_3 + k_4) x_3 - k_3 x_2 - k_4 x_1 &= f_3(t)
 \end{aligned}$$

In matrix form the equations become,

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} c_{v1} & 0 & 0 \\ 0 & c_{v2} & -c_{v2} \\ 0 & -c_{v2} & c_{v2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 + k_4 & -k_2 & -k_4 \\ -k_2 & k_1 + k_2 & -k_3 \\ -k_4 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}$$

Equations of Motion from Direct Matrix Formation

Observing the above coefficient matrices, we found that all diagonal terms are positive and contain terms that are directly attached to the corresponding elements.

Furthermore, all non-diagonal terms are *negative* and *symmetric*. They are symmetric since they are attached to two elements and the effects are the same in these two elements (a condition known as [Maxwell's Reciprocity Theorem](#)). They are negative due to the *relative* displacements/velocities of the two attached elements.

In summary,

1. Determine the number of degrees of freedom for the problem; this determines the size of the mass, damping, and stiffness matrices. Typically, one degree of freedom can be associated with each mass.
2. Enter the mass values (if associated with a degree of freedom) into the diagonals of the mass matrix; the exact ordering does not matter. All other values in the mass matrix are zero.

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

3. For each mass (associated with a degree of freedom), sum the damping from all dashpots attached to that mass; enter this value into the damping matrix at the diagonal location corresponding to that mass in the mass matrix.

$$C = \begin{bmatrix} c_{v1} & ? & ? \\ ? & c_{v2} & ? \\ ? & ? & c_{v2} \end{bmatrix}$$

4. Identify dashpots that are attached to two masses; label the masses as m and n . Write down the **negative** dashpot damping at the (m, n) and (n, m) locations in the damping matrix. Repeat for all dashpots. Any remaining terms in the damping matrix are zero.

$$C = \begin{bmatrix} c_{v1} & 0 & 0 \\ 0 & c_{v2} & -c_{v2} \\ 0 & -c_{v2} & c_{v2} \end{bmatrix}$$

5. For each mass (associated with a degree of freedom), sum the stiffness from all springs attached to that mass; enter this value into the stiffness matrix at the diagonal location corresponding to that mass in the mass matrix.

$$K = \begin{bmatrix} k_1 + k_2 + k_4 & ? & ? \\ ? & k_2 + k_3 & ? \\ ? & ? & k_3 + k_4 \end{bmatrix}$$

6. Identify springs that are attached to two masses; label the masses as m and n . Write down the **negative** spring stiffness at the (m, n) and (n, m) locations in the stiffness matrix. Repeat for all springs. Any remaining terms in the stiffness matrix are zero.

$$K = \begin{bmatrix} k_1 + k_2 + k_4 & -k_2 & -k_4 \\ -k_2 & k_2 + k_3 & -k_3 \\ -k_4 & -k_3 & k_3 + k_4 \end{bmatrix}$$

7. Sum the external forces applied on each mass (associated with a degree of freedom); enter this value into the force vector at the row location corresponding to the row location for that mass (in the mass matrix).

$$F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

8. The resulting matrix equation of motion is,

$$M\ddot{X} + C\dot{X} + KX = F$$

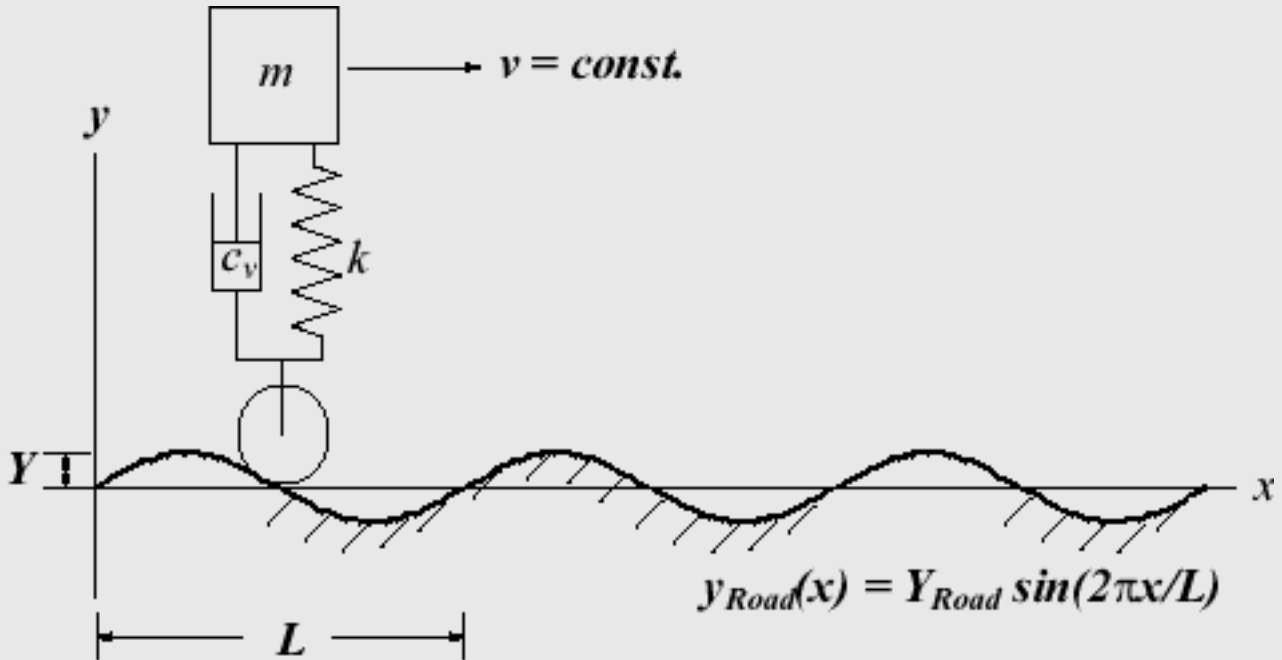
$$\Rightarrow \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} c_{v1} & 0 & 0 \\ 0 & c_{v2} & -c_{v2} \\ 0 & -c_{v2} & c_{v2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} +$$

$$\begin{bmatrix} k_1 + k_2 + k_4 & -k_2 & -k_4 \\ -k_2 & k_1 + k_2 & -k_3 \\ -k_4 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}$$

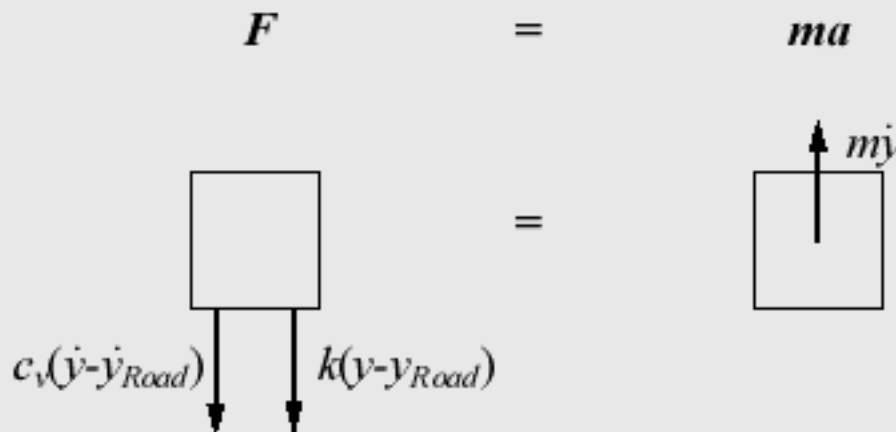
Dynamics: Moving Vehicle Example

Vehicle Traveling over a Bumpy Road

Consider a simple model of a vehicle moving over a bumpy road as illustrated in the following figure. Assume that the vehicle vibrates only in the vertical direction, the stiffness and damping effects of the tire can be neglected, and the tire has good traction and never leaves the road surface.



The free body diagram of this moving-base system can be illustrated as,



The equation of motion is thus,

$$m\ddot{y} + c_v(\dot{y} - \dot{y}_{Road}) + k(y - y_{Road}) = 0$$

Suppose that the vehicle is traveling at a constant speed, v , and the road roughness can be approximated by the equation,

$$y_{Road}(x) = Y_{Road} \sin\left(\frac{2\pi x}{L}\right)$$

The road roughness can then be rewritten in terms of time (instead of position),

$$y_{Road}(t) = Y_{Road} \sin\left(\frac{2\pi v}{L}t\right) = Y_{Road} \sin \omega t$$

$$\text{where } \omega = \frac{2\pi v}{L}$$

The harmonic moving base system is then equivalent to a harmonic vibration excitation with the equation of motion,

$$m\ddot{y} + c_v\dot{y} + ky = c_v\omega Y_{Road} \cos \omega t + kY_{Road} \sin \omega t$$

Since we seek the steady state solution for this problem (there are no "initial conditions" to prescribe), the displacement solution is just the [particular solution](#) for this problem,

$$x_p = Y_{Road} \sqrt{\frac{k^2 + (c_v\omega)^2}{(k - m\omega^2)^2 + (c_v\omega)^2}} \cos(\omega t - \alpha - \beta)$$

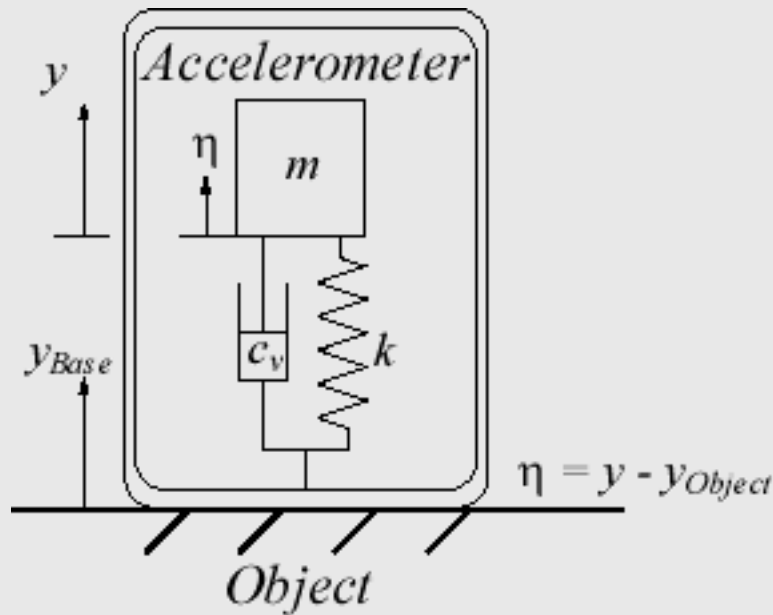
$$\text{where } \alpha = \tan^{-1}\left(\frac{c_v\omega}{k - m\omega^2}\right), \beta = \tan^{-1}\left(\frac{k}{c_v\omega}\right)$$

Note that if we had initial conditions, then we would need to also find the complimentary solution and weight the sum of the complimentary and particular solutions such that the initial conditions were satisfied. However, due to the damping in this system, the complimentary solution would die away exponentially and after a period of time only the particular solution (i.e. steady state solution) would remain.

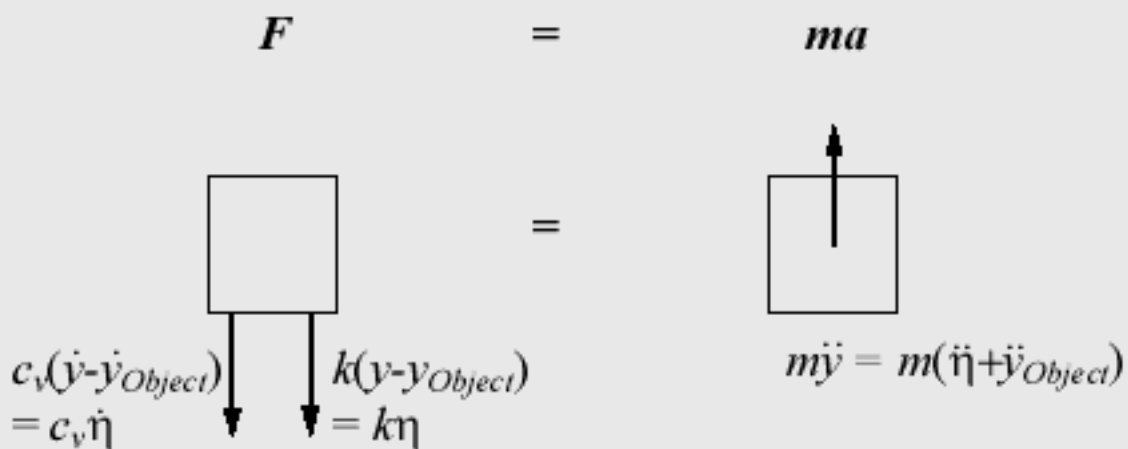
Dynamics: Accelerometer Example

Accelerometer

An accelerometer attached to a larger object can be modeled as a single degree-of-freedom vibration system excited by a moving base.



The above accelerometer model can be analyzed by the following free body diagram,



The equation of motion then becomes,

$$m\ddot{\eta} + c_v\dot{\eta} + k\eta = -m\ddot{y}_{Object}$$

We assume that the object is under the harmonic excitation, $y_{Object}(t) = Y_{Object} \sin \omega t$ to simplify the forcing function. The equation of motion becomes,

$$m\ddot{\eta} + c_v\dot{\eta} + k\eta = m\omega^2 Y_{Object} \sin \omega t$$

Accelerator Displacement Solution

The steady state solution for the accelerometer is found to be,

$$x_{steady}(t) = H \sin(\omega t - \varphi)$$

where the amplitude H , the phase φ , the damping ratio ζ , and the natural frequency ω_n are given by,

$$H = \frac{\left(\frac{\omega}{\omega_n}\right)^2 Y_{Object}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}}$$

$$\varphi = \tan^{-1} \left[\frac{2\zeta \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right]$$

$$\zeta = \frac{c_v}{2m\omega_n}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

Most accelerometers are constructed with a small mass and a short stiff spring, such that the natural frequency ω_n is much higher than the working frequency ω . As a result, the denominator of the amplitude H is approximately 1,

$$\frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}} \approx 1$$

This is important because the accelerometer can now track the acceleration of the target object directly, without the need for any amplitude corrections. To see this, compare the simplified displacement of the accelerometer with the acceleration amplitude of the target object, A_{Object} ,

$$x_{steady}(t) \approx \left(\frac{\omega}{\omega_n}\right)^2 Y_{Object} \sin(\omega t - \varphi)$$

$$A_{Object} = -\omega^2 Y_{Object}$$

Observe that the target object's acceleration amplitude is contained within the accelerometer's displacement directly,

$$\eta_{steady}(t) \approx -\frac{A_{Object}}{\omega_n^2} \sin(\omega t - \varphi)$$

For $\zeta = 0.707$, the effective frequency range can be up to $0.4 \omega_n$ with less than 1% error. In fact, the results are often acceptable up to $0.6 \omega_n$ without adjustment.

For modern piezoelectric accelerometers, the damping ratio is close to zero. In addition, their mass is very small (approx. 10 grams; less than 1 oz) and they have a very high stiffness, resulting in natural frequencies of 30 kHz or more. Hence their working range can extend up to 5 kHz or higher.

Seismometer

In contrast to small accelerometers, bulky transducers with large masses and soft springs would have very small natural frequencies. If the natural frequency is much smaller than the working frequency, then the amplitude H from above simplifies even more,

$$\frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}} \approx 1$$

The displacement of the object can then be obtained directly (adjusted for a phase delay),

$$\eta(t) \approx Y_{Object} \sin(\omega t - \varphi)$$

This kind of transducer is used as a seismometer to detect earthquakes, or as vibrometers to measure vibration displacements. For $\zeta = 0.707$, the effective frequency range can be as low as $3 \omega_n$.

Fluid Mechanics: Overview

Fluid Preliminaries

By definition, a fluid is a material continuum that is unable to withstand a static shear stress. Unlike an elastic solid which responds to a shear stress with a recoverable deformation, a fluid responds with an irrecoverable flow.

Variables needed to define a fluid and its environment are:

Quantity	Symbol	Object	Units
pressure	p	scalar	N/m ²
velocity	\mathbf{v}	vector	m/s
density	ρ	scalar	kg/m ³
viscosity	μ	scalar	kg/m-s
body force	\mathbf{b}	vector	N/kg
time	t	scalar	s

Examples of fluids include gases and liquids. Typically, liquids are considered to be incompressible, whereas gases are considered to be compressible. However, there are exceptions in everyday engineering applications.

Types of Flow; Reynolds Number

Fluid flow can be either [laminar](#) or [turbulent](#). The factor that determines which type of flow is present is the ratio of inertia forces to viscous forces within the fluid, expressed by the nondimensional Reynolds Number,

$$R = \frac{\rho V D}{\mu}$$

where V and D are a fluid characteristic velocity and distance. For example, for fluid flowing in a pipe, V could be the average fluid velocity, and D would be the pipe diameter.

Typically, viscous stresses within a fluid tend to stabilize and organize the flow, whereas excessive fluid inertia tends to disrupt organized flow leading to chaotic turbulent behavior.

Fluid flows are laminar for Reynolds Numbers up to 2000. Beyond a Reynolds Number of 4000, the flow is completely turbulent. Between 2000 and 4000, the flow is in transition between laminar and turbulent, and it is possible to find subregions of both flow types within a given flow field.

Governing Equations

Laminar fluid flow is described by the [Navier-Stokes](#) equations. For cases of inviscid flow, the [Bernoulli](#) equation can be used to describe the flow. When the flow is zero (i.e. statics), the fluid is governed by the laws of [fluid statics](#).

Fluid Mechanics: Navier Stokes

Navier-Stokes Equations

The motion of a non-turbulent, Newtonian fluid is governed by the Navier-Stokes equation:

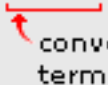
$$-\vec{\nabla}p + \mu \left(\vec{\nabla}^2 \mathbf{v} \right) + \frac{1}{3} \mu \left(\vec{\nabla} \left(\vec{\nabla} \cdot \mathbf{v} \right) \right) + \rho \mathbf{b} = \rho \dot{\mathbf{v}} \quad \text{compressible fluid}$$

$$-\vec{\nabla}p + \mu \left(\vec{\nabla}^2 \mathbf{v} \right) + \rho \mathbf{b} = \rho \dot{\mathbf{v}} \quad \text{incompressible fluid}$$

The above equation can also be used to model turbulent flow, where the fluid parameters are interpreted as time-averaged values.

The time-derivative of the fluid velocity in the Navier-Stokes equation is the *material derivative*, defined as:

$$\dot{\mathbf{v}} \triangleq \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$$



convection term

The material derivative is distinct from a normal derivative because it includes a convection term, a very important term in fluid mechanics. This unique derivative will be denoted by a "dot" placed above the variable it operates on.

Navier-Stokes Background

On the most basic level, laminar (or time-averaged turbulent) fluid behavior is described by a set of fundamental equations. These equations are:

$$\dot{\rho} + \rho \left(\vec{\nabla} \cdot \mathbf{v} \right) = 0 \quad \text{fluid continuity}$$

$$\vec{\nabla} \cdot \underline{\underline{\mathbf{T}}} + \rho \mathbf{b} = \rho \dot{\mathbf{v}} \quad \text{equation of motion}$$

$$\underline{\underline{\mathbf{T}}} = -p \underline{\underline{\mathbf{I}}} - \frac{2}{3} \mu \left(\text{tr} \underline{\underline{\mathbf{D}}} \right) \underline{\underline{\mathbf{I}}} + 2\mu \underline{\underline{\mathbf{D}}} \quad \text{constitutive relation}$$

$$\underline{\underline{\mathbf{D}}} = \frac{1}{2} \left(\mathbf{v} \vec{\nabla} + \vec{\nabla} \mathbf{v} \right) \quad \text{fluid kinematics}$$

The Navier-Stokes equation is obtained by combining the fluid kinematics and constitutive relation into the fluid equation of motion, and eliminating the parameters $\underline{\underline{\mathbf{D}}}$ and $\underline{\underline{\mathbf{T}}}$. These terms are defined below:

Quantity	Symbol	Object	Units
fluid stress	T	2 nd order tensor	N/m ²
strain rate	D	2 nd order tensor	1/s
unity tensor	I	2 nd order tensor	1

Fluid Mechanics: Bernoulli

Bernoulli Equation

A [non-turbulent](#), [perfect](#), [compressible](#), and [barotropic](#) fluid undergoing steady motion is governed by the Bernoulli Equation:

$$\frac{V^2}{2g} + z + \frac{\tilde{P}}{g} = C \text{ (streamline)}$$

where g is the gravity acceleration constant (9.81 m/s²; 32.2 ft/s²), V is the velocity of the fluid, and z is the height above an arbitrary datum. C remains constant along any [streamline](#) in the flow, but varies from streamline to streamline. If the flow is [irrotational](#), then C has the same value for all streamlines.

The function \tilde{P} is the "[pressure per density](#)" in the fluid, and follows from the barotropic equation of state, $p = p(\rho)$.

For an [incompressible](#) fluid, the function \tilde{P} simplifies to p/ρ , and the incompressible Bernoulli Equation becomes:

$$\frac{V^2}{2g} + z + \frac{p}{\rho g} = C$$

Derivation from Navier-Stokes

The [Navier-Stokes](#) equation for a perfect fluid reduce to the **Euler Equation**:

$$-\vec{\nabla}p + \rho \mathbf{b} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \vec{\nabla} \mathbf{v} \right)$$

Rearranging, and assuming that the body force \mathbf{b} is due to gravity only, we can eventually integrate over space to remove any vector derivatives,

$$\begin{aligned}
 -\frac{1}{\rho} \vec{\nabla} p - g \mathbf{i}_z &= \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \vec{\nabla} \mathbf{v} \\
 -\frac{d\tilde{P}}{dp} \vec{\nabla} p - \vec{\nabla}(gz) &= \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \vec{\nabla} \mathbf{v} \\
 0 &= \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \vec{\nabla} \mathbf{v} + \vec{\nabla}(\tilde{P} + gz) \\
 C(t, \text{streamline}) &= \int \frac{\partial v_i}{\partial t} dx_i + \frac{1}{2} V^2 + \tilde{P} + gz
 \end{aligned}$$

If the fluid motion is also steady (implying that all derivatives with respect to time are zero), then we arrive at the Bernoulli equation after dividing out by the gravity constant (and absorbing it into the constant C),

$$\frac{V^2}{2g} + z + \frac{\tilde{P}}{g} = C(\text{streamline})$$

Note that the fluid's barotropic nature allowed the following chain rule application,

$$\vec{\nabla} \tilde{P} = \frac{d\tilde{P}}{dp} \vec{\nabla} p = \frac{1}{\rho} \vec{\nabla} p$$

with the "**pressure per density**" function \tilde{P} defined as,

$$\tilde{P}(p) = \int_{p_0}^p \frac{dp}{\rho}$$

Fluid Mechanics: Fluid Statics

Fluids at Rest

A [barotropic](#), [compressible](#) fluid at rest is governed by the statics equation,

$$\Delta p = -g \int_{z_0}^z \rho dz$$

where z is the height above an arbitrary datum, and g is the gravity acceleration constant (9.81 m/s²; 32.2 ft/s²). This equation describes the pressure profile of the atmosphere, for example.

For an incompressible fluid, the statics equation simplifies to,

$$\Delta p = -\rho g \Delta z$$

This equation describes the pressure profile in a body of water, or in a manometer.

If the fluid is compressible but [barotropic](#), then the density and the pressure can be integrated into the "[pressure per density](#)" function \check{P} , giving the following alternate form for the compressible fluid statics equation,

$$\check{P}(p) = -g \Delta z$$

Note that the equation at the top of the page can still be applied though, as it makes no assumption on the fluid's equation of state.

Derivation from Navier-Stokes

The [Navier-Stokes](#) equation for a fluid at rest reduce to,

$$-\nabla p + \rho \mathbf{b} = 0$$

Rearranging, and assuming that the body force \mathbf{b} is due to gravity only, we can integrate over space to remove any vector derivatives,

$$\vec{\nabla} p = -\rho g \mathbf{i}_z$$

$$p(z) = p(z_0) - g \int_{z_0}^z \rho d\tilde{z}$$

For the barotropic fluid case, the derivation can be repeated in a fashion similar to that of [Bernoulli](#),

$$-\frac{1}{\rho} \vec{\nabla} p - g \mathbf{i}_z = 0$$

$$-\frac{d\tilde{P}}{dp} \vec{\nabla} p - \vec{\nabla}(gz) = 0$$

$$\vec{\nabla}(\tilde{P} + gz) = 0$$

$$\tilde{P} + g\Delta z = 0$$

Fluid Mechanics: Glossary

Glossary

barotropic	A barotropic fluid is one whose pressure and density are related by an equation of state that does not contain the temperature as a dependent variable. Mathematically, the equation of state can be expressed as $p = p(\rho)$ or $\rho = \rho(p)$.
compressible	A fluid flow is compressible if its density ρ changes appreciably (typically by a few percent) within the domain of interest. Typically, this will occur when the fluid velocity exceeds Mach 0.3. Hence, low velocity flows (both gas and liquids) behave incompressibly.
density, ρ	The mass of fluid per unit volume. For a compressible fluid flow, the density can vary from place to place.
incompressible	An incompressible fluid is one whose density is constant everywhere. All fluids behave incompressibly (to within 5%) when their maximum velocities are below Mach 0.3.
inviscid	Not viscous.
irrotational	An irrotational fluid flow is one whose streamlines never loop back on themselves. Typically, only inviscid fluids can be irrotational. Of course, a uniform viscid fluid flow without boundaries is also irrotational, but this is a special (and boring!) case.
laminar (non-turbulent)	An organized flow field that can be described with streamlines. In order for laminar flow to be permissible, the viscous stresses must dominate over the fluid inertia stresses.
Mach	Mach number is the relative velocity of a fluid compared to its sonic velocity. Mach numbers less than 1 correspond to sub-sonic velocities, and Mach numbers > 1 correspond to super-sonic velocities.
Newtonian	A Newtonian fluid is a viscous fluid whose shear stresses are a linear function of the fluid strain rate. Mathematically, this can be expressed as: $\tau_{ij} = K_{ijqp} * D_{pq}$, where τ_{ij} is the shear stress component, and D_{pq} are fluid strain rate components.
perfect	A perfect fluid is defined as a fluid with zero viscosity (i.e. inviscid).

rotational	A rotational fluid flow can contain streamlines that loop back on themselves. Hence, fluid particles following such streamlines will travel along closed paths. Bounded (and hence nonuniform) viscous fluids exhibit rotational flow, typically within their boundary layers. Since all real fluids are viscous to some amount, all real fluids exhibit a level of rotational flow somewhere in their domain. Regions of rotational flow correspond to the regions of viscous losses in a fluid. Inviscid fluid flows can also be rotational, but these are special nonphysical cases. For an inviscid fluid flow to be rotational, it must be set up that way by initial conditions. The amount of rotation (called the <i>velocity circulation</i>) in an inviscid fluid flow is conserved, provided that the fluid is also barotropic and subject only to conservative body forces. This conservation is known as <i>Kelvin's Theorem</i> of constant circulation.
Stokesian	A Stokesian (or non-Newtonian) fluid is a viscous fluid whose shear stresses are a non-linear function of the fluid strain rate.
streamline	A path in a steady flow field along which a given fluid particle travels.
turbulent	A flow field that cannot be described with streamlines in the absolute sense. However, time-averaged streamlines can be defined to describe the average behavior of the flow. In turbulent flow, the inertia stresses dominate over the viscous stresses, leading to small-scale chaotic behavior in the fluid motion.
viscosity, μ	A fluid property that relates the magnitude of fluid shear stresses to the fluid strain rate, or more simply, to the spatial rate of change in the fluid velocity field. Mathematically, this is expressed as: $\tau = \mu^*(dV/dy)$, where τ is the shear stress in the same direction as the fluid velocity V , and y is a direction perpendicular to the fluid velocity direction.

Heat Transfer: Introduction

Basics of Heat Transfer

In the simplest of terms, the discipline of heat transfer is concerned with only two things: **temperature**, and the **flow of heat**. Temperature represents the amount of thermal energy available, whereas heat flow represents the movement of thermal energy from place to place.

On a microscopic scale, thermal energy is related to the kinetic energy of molecules. The greater a material's temperature, the greater the thermal agitation of its constituent molecules (manifested both in linear motion and vibrational modes). It is natural for regions containing greater molecular kinetic energy to pass this energy to regions with less kinetic energy.

Several material properties serve to modulate the heat transferred between two regions at differing temperatures. Examples include thermal conductivities, specific heats, material densities, fluid velocities, fluid viscosities, surface emissivities, and more. Taken together, these properties serve to make the solution of many heat transfer problems an involved process.

Heat Transfer Mechanisms

Heat transfer mechanisms can be grouped into 3 broad categories:

Conduction: Regions with greater molecular kinetic energy will pass their thermal energy to regions with less molecular energy through direct molecular collisions, a process known as conduction. In metals, a significant portion of the transported thermal energy is also carried by conduction-band electrons.

Convection: When heat conducts into a static fluid it leads to a local volumetric expansion. As a result of gravity-induced pressure gradients, the expanded fluid parcel becomes buoyant and displaces, thereby transporting heat by fluid motion (i.e. convection) in addition to conduction. Such heat-induced fluid motion in initially static fluids is known as **free convection**.

For cases where the fluid is already in motion, heat conducted into the fluid will be transported away chiefly by fluid convection. These cases, known as **forced convection**, require a pressure gradient to drive the fluid motion, as opposed to a gravity gradient to induce motion through buoyancy.

Radiation: All materials radiate thermal energy in amounts determined by their temperature, where the energy is carried by photons of light in the infrared and visible portions of the electromagnetic spectrum. When temperatures are uniform, the radiative flux between objects is in equilibrium and no net thermal energy is exchanged. The balance is upset when temperatures are not uniform, and thermal energy is transported from surfaces of higher to surfaces of lower temperature.

Heat Transfer: Conduction

Fourier Law of Heat Conduction

When there exists a temperature gradient within a body, heat energy will flow from the region of high temperature to the region of low temperature. This phenomenon is known as conduction heat transfer, and is described by **Fourier's Law** (named after the French physicist Joseph Fourier),

$$\mathbf{q} = -k\vec{\nabla}T$$

This equation determines the [heat flux](#) vector \mathbf{q} for a given temperature profile T and [thermal conductivity](#) k . The minus sign ensures that heat flows down the temperature gradient.

Heat Equation (Temperature Determination)

The temperature profile within a body depends upon the rate of its internally-generated heat, its capacity to store some of this heat, and its rate of thermal conduction to its boundaries (where the heat is transferred to the surrounding environment). Mathematically this is stated by the **Heat Equation**,

$$\vec{\nabla}^2 T - \frac{1}{\alpha} \frac{\partial T}{\partial t} = -\frac{1}{k} q_{gen}$$

along with its boundary conditions, equations that prescribe either the temperature T on, or the heat flux q through, all of the body boundaries Ω ,

$$T(\Omega_a) = T_{prescribed}$$

$$q(\Omega_b) = q_{prescribed}$$

$$\Omega_a \cup \Omega_b = \Omega$$

In the Heat Equation, the [power generated per unit volume](#) is expressed by q_{gen} . The [thermal diffusivity](#) α is related to the [thermal conductivity](#) k , the [specific heat](#) c , and the [density](#) ρ by,

$$\alpha = \frac{k}{\rho c}$$

For **Steady State** problems, the Heat Equation simplifies to,

$$\vec{\nabla}^2 T = -\frac{1}{k} q_{gen}$$

Derivation of the Heat Equation

The heat equation follows from the **conservation of energy** for a small element within the body,

$$\begin{array}{ccccccc} \text{heat conducted} & + & \text{heat generated} & = & \text{heat conducted} & + & \text{change in} \\ \text{in} & & \text{within} & & \text{out} & & \text{energy} \\ & & & & & & \text{stored} \\ & & & & & & \text{within} \end{array}$$

Mathematically, this equation is expressed as,

$$\vec{\nabla} \cdot \mathbf{q} = q_{gen} - \frac{de}{dt}$$

The change in [internal energy](#) e is related to the body's ability to store heat by raising its temperature, given by,

$$\frac{de}{dt} = \rho c \frac{dT}{dt}$$

One can substitute for \mathbf{q} using Fourier's Law of heat conduction from above to arrive at the Heat Equation,

$$\begin{aligned} \vec{\nabla} \cdot (-k \vec{\nabla} T) &= q_{gen} - \rho c \frac{dT}{dt} \\ -k \vec{\nabla}^2 T + \rho c \frac{\partial T}{\partial t} &= q_{gen} \\ \vec{\nabla}^2 T - \frac{1}{\alpha} \frac{\partial T}{\partial t} &= -\frac{1}{k} q_{gen} \end{aligned}$$

Heat Transfer: 1-Dimensional Heat Conduction

Steady State 1-Dimensional Heat Conduction

For problems where the temperature variation is only **1-dimensional** (say, along the x -coordinate direction), [Fourier's Law](#) of heat conduction simplifies to the scalar equations,

$$q = -k \frac{\partial T}{\partial x} \qquad \dot{Q} = -kA \frac{\partial T}{\partial x}$$

where the [heat flux](#) q depends on a given temperature profile T and [thermal conductivity](#) k . The minus sign ensures that heat flows down the temperature gradient.

In the above equation on the right, \dot{Q} represents the heat flow through a defined cross-sectional area A , measured in watts,

$$\dot{Q} = \int_A q \cdot dA$$

Integrating the 1D heat flow equation through a material's thickness Δx gives,

$$\dot{Q} = \frac{kA}{\Delta x} (T_1 - T_2)$$

where T_1 and T_2 are the temperatures at the two boundaries.

The R-Value in Insulation

In general terms, heat transfer is quantified by Newton's Law of Cooling,

$$\dot{Q} = hA \cdot \Delta T$$

where h is the heat transfer coefficient. For conduction, h is a function of the thermal conductivity and the material thickness,

$$h = \frac{k}{\Delta x}$$

In words, h represents the heat flow per unit area per unit temperature difference. The larger h is, the larger the heat transfer Q .

The inverse of h is commonly defined as the *R-value*,

$$R = \frac{1}{h} = \frac{\Delta x}{k}$$

The R -value is used to describe the effectiveness of insulations, since as the inverse of h , it represents the resistance to heat flow. The larger the R , the less the heat flow \dot{Q} .

R is often expressed in imperial units when listed in tables. Conversion to SI-units is provided in the [Units Section](#).

To convert R into a thermal conductivity k , we must divide the thickness of the insulation by the R value (or just solve for k from the above equation),

$$k = \frac{\Delta x}{R}$$

Heat Transfer: Electrical Analogy

Electrical Analogy for 1D Heat Conduction

By comparing the steady state heat flow equation with Ohm's Law for current flow through a resistor, we see that they have similar forms,

$$\dot{Q} = \frac{kA}{\Delta x}(T_1 - T_2) \quad \Leftrightarrow \quad I = \frac{1}{R}(V_1 - V_2)$$

We can therefore draw the following analogies:

Heat Flow, \dot{Q}

\Leftrightarrow

Current, I

Temperature Difference, $T_1 - T_2$

\Leftrightarrow

Voltage Difference, $V_1 - V_2$

Thermal Resistance, $R_T = \Delta x/k^*A$

\Leftrightarrow

Electrical Resistance, R

The electrical to heat conduction analogy allows one to apply laws from circuit theory to solve more complicated conduction problems, such as the heat flow through conducting layers attached in parallel or series.

Heat Transfer: Symbols

Symbols

Notation and symbols commonly used in [conduction theory](#) are summarized below:

Independent Parameters

Quantity	Symbol	Object	Units
temperature	T	scalar	K
time	t	scalar	s
thermal conductivity	k	scalar	W/m-K
specific heat	c	scalar	J/kg-K
density	ρ	scalar	kg/m ³
internal heat generation	q_{gen}	scalar	W/m ³

Dependent Parameters

Quantity	Symbol	Object	Units
thermal diffusivity	α	scalar	m ² /s
heat flux	\mathbf{q}	vector	W/m ²
1-D heat flux	q	scalar	W/m ²
heat transfer rate	\dot{Q}	scalar	W
total heat	Q	scalar	J

A Note on the q 's

Confusion can arise over the notation used to describe heat flux, total heat flow, and total heat, especially since they all involve some permutation of the letter q .

In an effort to fully clarify the usage of q here, please note the following interdependencies, explained both with math and with words:

Mathematical Relationships

	\dot{Q}	q	q_{gen}
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Q	$Q = \int_{t_1}^{t_2} \dot{Q} \cdot dt$	$Q = \int_{t_1}^{t_2} \int_A q \cdot dA \cdot dt$	$Q = \int_{t_1}^{t_2} \oint_V q_{gen} \cdot dV \cdot dt$
\dot{Q}		$\dot{Q} = \int_A q \cdot dA$	$\dot{Q} = \oint_V q_{gen} \cdot dV$
q			$\frac{dq}{dx} = q_{gen} - \frac{de}{dt}$

Interpretations

	\dot{Q}	q	q_{gen}
Q	Total heat transferred is the heat transfer rate summed over the time interval.	Total heat transferred through area A is the heat flux integrated over A and the time interval.	Total heat generated in volume V is the heat generation rate integrated over V and the time interval.
\dot{Q}		Heat transfer rate across an area A is the heat flux integrated over A .	Steady-state heat transfer rate out of a volume V is the heat generation rate integrated over V .
q			One-dimensional version of the conservation of energy statement, where e is the internal energy density reflected in the bodies temperature.

Heat Transfer: Convection

Introduction

Heat energy transfers between a solid and a fluid when there is a temperature difference between the fluid and the solid. This is known as "convection heat transfer". Generally, convection heat transfer can not be ignored when there is a significant fluid motion around the solid.

The temperature of the solid due to an external field such as fluid buoyancy can induce a fluid motion. This is known as "natural convection" and it is a strong function of the temperature difference between the solid and the fluid. Blowing air over the solid by using external devices such as fans and pumps can also generate a fluid motion. This is known as "forced convection".

Fluid mechanics plays a major role in determining convection heat transfer. For each kind of convection heat transfer, the fluid flow can be either laminar or turbulent. Laminar flow generally occurs in relatively low velocities in a smooth laminar boundary layer over smooth small objects, while turbulent flow forms when the boundary layer is shedding or breaking due to higher velocities or rough geometries.

Heat Transfer: Non-Dimensional Parameters

Non-Dimensional Parameters

As is common with fluid mechanics analysis, a number of dimensionless parameters are employed to describe convective heat transfer. A summary of these variables is included in the following tables:

General Convection (Forced and Free)

Parameter	Formula	Interpretation
Prandtl Number:	$Pr = \frac{\nu}{\alpha} = \frac{c_p \mu}{k}$	Ratio of fluid velocity boundary layer thickness to the fluid temperature boundary layer thickness.
Nusselt Number:	$Nu = \frac{hL}{k}$	Ratio of heat transferred from surface to heat conducted away by fluid.

Forced Convection Only

Parameter	Formula	Interpretation
Reynolds Number:	$Re_L = \frac{u_\infty L}{\nu} = \frac{\rho u_\infty L}{\mu}$	Ratio of fluid inertia stress to viscous stress (for flow over flat plates).
Reynolds Number:	$Re_D = \frac{u_\infty D}{\nu} = \frac{\rho u_\infty D}{\mu}$	(Reynolds Number for pipe flow).
Stanton Number:	$St = \frac{h}{\rho c_p u_\infty} = \frac{Nu}{Re \cdot Pr}$	

Free Convection Only

Parameter	Formula	Interpretation
Grashof Number:	$Gr = \frac{g \beta \Delta T L^3}{\nu^2}$	Ratio of fluid buoyancy stress to viscous stress.
Rayleigh Number:	$Ra = Gr \cdot Pr$	

Heat Transfer: Forced Convection

Newton's Law of Cooling

The essential ingredients of forced convection heat transfer analysis are given by **Newton's Law of Cooling**,

$$\dot{Q} = hA(T_w - T_\infty) = hA \cdot \Delta T$$

The rate of heat \dot{Q} transferred to the surrounding fluid is proportional to the object's exposed area A , and the difference between the object temperature T_w and the fluid free-stream temperature T_∞ .

The constant of proportionality h is termed the *convection heat-transfer coefficient*. Other terms describing h include *film coefficient* and *film conductance*.

Definition of Symbols

Independent Parameters for the Fluid

Quantity	Symbol	Object	Units
free-stream temperature	T_∞	scalar	K
free-stream velocity	u_∞	scalar	m/s
kinematic viscosity	ν	scalar	m ² /s
dynamic viscosity	μ	scalar	kg/m-s
density	ρ	scalar	kg/m ³
thermal diffusivity	α	scalar	m ² /s
specific heat	c_p	scalar	J/kg-K
thermal conductivity	k	scalar	W/m-K

Independent Parameters for the Object

Quantity	Symbol	Object	Units
surface reference length	L	scalar	m
surface diameter (for pipes)	D	scalar	m
surface (wall) temperature	T_w	scalar	K

Dependent Parameters

Quantity	Symbol	Object	Units
surface to fluid temp. difference	ΔT	scalar	K
heat transfer coefficient	h	scalar	W/m ² -K

Heat Transfer: Flow Profile Over a Flat Plate

Flow Profile Over a Flat Plate

Two-dimensional flow analysis over a flat plate serves well to illustrate several key concepts in forced convection heat transfer.

The viscosity of the fluid requires that the fluid have zero velocity at the plate's surface. As a result a boundary layer exists where the fluid velocity changes from u_∞ in the free stream (far from the plate) to zero at the plate. Within this boundary layer, the flow is initially laminar but can proceed to turbulence once the Reynolds Number Re of the flow is sufficiently high. The transition from laminar to turbulent for flow over a flat plate occurs in the range,

$$3 \times 10^5 < Re_x < 3 \times 10^6, \quad Re_x = \frac{\rho u_\infty x}{\mu}$$

For the current problem, we consider that the plate is heated starting at a point $x = x_0$ and continuing downstream. Furthermore we assume that the plate is maintained at constant temperature T_w , making this problem **isothermal**. We are interested only in **laminar** flow, so it is assumed that the plate length L is sufficiently short such that turbulent flow is never triggered (i.e. $Re_x < 3 \times 10^5$).

Nusselt Number Calculation

An analysis of the fluid flow over the plate that considers conservation of momentum and energy, including the effects of viscosity, temperature, and heat entering the fluid from the plate, results in the following equation for the Nusselt Number Nu as a function of x ,

$$Nu(x) = 0.332 \frac{(Pr)^{1/3} \cdot \sqrt{Re(x)}}{\left(1 - \left(\frac{x_0}{x}\right)^{3/4}\right)^{1/3}}$$

The Nusselt Number is a non-dimensional ratio of the heat entering the fluid from the surface to the heat conducted away by the fluid. It is defined by the equation,

$$Nu(x) = \frac{h(x) \cdot x}{k}$$

The Prandtl Number Pr is a non-dimensional ratio of the viscous boundary layer thickness to the thermal boundary layer thickness. It is defined by the equation,

$$Pr = \frac{\nu}{\alpha} = \frac{c_p \mu}{k}$$

If the plate (of length L) is uniformly heated over its entirety ($x_0 = 0$), then the average Nusselt Number is found to be,

$$Nu_L = 0.664 \cdot (Pr)^{1/3} \cdot \sqrt{Re_L}$$

Heat Transfer Calculation

The Nusselt Number equation can be used to calculate the heat transfer coefficient h via,

$$h_{plate} = \frac{k}{L} Nu_L$$

which can then be used to calculate the heat convected away by the fluid via Newton's Law of cooling,

$$\dot{Q} = h_{plate} A (T_w - T_\infty)$$

$$\dot{Q}_{plate} = 0.664 \cdot (Pr)^{1/3} \cdot \sqrt{Re_L} \cdot \frac{kA}{L} \Delta T$$

Since fluid properties (such as viscosity, diffusivity, etc.) can vary significantly with temperature, there can be some ambiguity as to which temperature one should use to select property values. The recommended approach is the use the average of the wall and free-stream temperatures, defined as the film temperature T_f ,

$$T_f = \frac{T_w + T_\infty}{2}$$

Heat Transfer: Free Convection

Newton's Law of Cooling

Similar to [forced convection](#), heat transfer due to free convection is described by **Newton's Law of Cooling**,

$$\dot{Q} = hA(T_w - T_\infty) = hA \cdot \Delta T$$

The rate of heat \dot{Q} transferred to the surrounding fluid is proportional to the object's exposed area A , and the difference between the object temperature T_w and the fluid free-stream temperature T_∞ .

The constant of proportionality h is termed the *convection heat-transfer coefficient*. Other terms describing h include *film coefficient* and *film conductance*.

Variables needed to parameterize free convection are:

Definition of Symbols

Independent Parameters for the Fluid

Quantity	Symbol	Object	Units
bulk temperature	T_∞	scalar	K
kinematic viscosity	ν	scalar	m ² /s
coef. of thermal expansion	β	scalar	1/K
dynamic viscosity	μ	scalar	kg/m-s
density	ρ	scalar	kg/m ³
thermal diffusivity	α	scalar	m ² /s
specific heat	c_p	scalar	J/kg-K
thermal conductivity	k	scalar	W/m-K

Independent Parameters for the Object

Quantity	Symbol	Object	Units
surface reference length	L	scalar	m
surface diameter (for pipes)	D	scalar	m
surface (wall) temperature	T_w	scalar	K

Dependent Parameters

Quantity	Symbol	Object	Units
surface to fluid temp. difference	ΔT	scalar	K
heat transfer coefficient	h	scalar	W/m ² -K

Heat Transfer: Radiation

Introduction

Radiation heat transfer is concerned with the exchange of thermal radiation energy between two or more bodies. Thermal radiation is defined as electromagnetic radiation in the wavelength range of 0.1 to 100 microns (which encompasses the visible light regime), and arises as a result of a temperature difference between 2 bodies.

No medium need exist between the two bodies for heat transfer to take place (as is needed by [conduction](#) and convection). Rather, the intermediaries are photons which travel at the speed of light.

The heat transferred into or out of an object by thermal radiation is a function of several components. These include its surface reflectivity, emissivity, surface area, temperature, and geometric orientation with respect to other thermally participating objects. In turn, an object's surface reflectivity and emissivity is a function of its surface conditions (roughness, finish, etc.) and composition.

Absorption and Emissivity

Radiation heat transfer must account for both incoming and outgoing thermal radiation.

Incoming radiation can be either absorbed, reflected, or transmitted. This decomposition can be expressed by the relative fractions,

$$1 = \varepsilon_{\text{reflected}} + \varepsilon_{\text{absorbed}} + \varepsilon_{\text{transmitted}}$$

Since most solid bodies are opaque to thermal radiation, we can ignore the transmission component and write,

$$1 = \varepsilon_{\text{reflected}} + \varepsilon_{\text{absorbed}}$$

To account for a body's outgoing radiation (or its *emissive power*, defined as the heat flux per unit time), one makes a comparison to a perfect body who emits as much thermal radiation as possible. Such an object is known as a [blackbody](#), and the ratio of the actual [emissive power](#) E to the emissive power of a blackbody is defined as the [surface emissivity](#) ε ,

$$\varepsilon = \frac{E}{E_{\text{blackbody}}}$$

By stating that a body's surface emissivity is equal to its absorption fraction, **Kirchhoff's Identity** binds incoming and outgoing radiation into a useful dependent relationship,

$$\varepsilon = \varepsilon_{\text{absorbed}}$$

Heat Transfer: Black Body Radiation

Blackbody Radiation Heat Transfer

The heat emitted by a blackbody (per unit time) at an absolute temperature of T is given by the **Stefan-Boltzmann Law** of thermal radiation,

$$\dot{Q} = A\sigma T^4 = AE_{blackbody}$$

where \dot{Q} has units of Watts, A is the total radiating area of the blackbody, and σ is the [Stefan-Boltzmann constant](#).

A small blackbody at absolute temperature T enclosed by a much larger blackbody at absolute temperature T_e will transfer a **net heat flow** of,

$$\dot{Q} = A\sigma(T^4 - T_e^4)$$

Why is this a "net" heat flow? The small blackbody still emits a total heat flow given by the Stefan-Boltzmann law. However, the small blackbody also receives and absorbs all the thermal energy emitted by the large enclosing blackbody, which is a function of its temperature T_e . The difference in these two heat flows is the net heat flow lost by the small blackbody.

Gray Body Radiation Heat Transfer

Bodies that emit less thermal radiation than a blackbody have [surface emissivities](#) ε less than 1. If the surface emissivity is independent of wavelength, then the body is called a "gray" body, in that no particular wavelength (or color) is favored.

The net heat transfer from a small gray body at absolute temperature T with surface emissivity ε to a much larger enclosing gray (or black) body at absolute temperature T_e is given by,

$$\dot{Q} = \varepsilon A\sigma(T^4 - T_e^4)$$

Heat Transfer: Radiation View Factors

Radiation View Factors

The above equations for blackbodies and graybodies assumed that the small body could see only the large enclosing body and nothing else. Hence, all radiation leaving the small body would reach the large body. For the case where two objects can see more than just each other, then one must introduce a [view factor](#) F and the heat transfer calculations become significantly more involved.

The view factor F_{12} is used to parameterize the fraction of thermal power leaving object 1 and reaching object 2. Specifically, this quantity is equal to,

$$\dot{Q}_{1 \rightarrow 2} = A_1 F_{12} \varepsilon_1 \sigma T_1^4$$

Likewise, the fraction of thermal power leaving object 2 and reaching object 1 is given by,

$$\dot{Q}_{2 \rightarrow 1} = A_2 F_{21} \varepsilon_2 \sigma T_2^4$$

The case of two blackbodies in thermal equilibrium can be used to derive the following **reciprocity** relationship for view factors,

$$A_1 F_{12} = A_2 F_{21}$$

Thus, once one knows F_{12} , F_{21} can be calculated immediately.

Radiation view factors can be analytically derived for simple geometries and are tabulated in several references on heat transfer (e.g. Holman, 1986). They range from zero (e.g. two small bodies spaced very far apart) to 1 (e.g. one body is enclosed by the other).

Heat Transfer Between Two Finite Graybodies

The heat flow transferred from Object 1 to Object 2 where the two objects see only a fraction of each other and nothing else is given by,

$$\dot{Q} = \left(\frac{1 - \varepsilon_1}{\varepsilon_1} + \frac{1}{F_{12}} + \left(\frac{1 - \varepsilon_2}{\varepsilon_2} \right) \frac{A_1}{A_2} \right)^{-1} A_1 \sigma (T_1^4 - T_2^4)$$

This equation demonstrates the usage of F_{12} , but it represents a non-physical case since it would be impossible to position two finite objects such that they can see only a portion of each other and "nothing" else. On the contrary, the complementary view factor $(1 - F_{12})$ cannot be neglected as radiation energy sent in those directions must be accounted for in the thermal bottom line.

A more realistic problem would consider the same two objects surrounded by a third surface that can absorb and readmit thermal radiation yet is non-conducting. In this manner, all thermal energy that is absorbed by this third surface will be readmitted; no energy can be removed from the system through this surface. The equation describing the heat flow from Object 1 to Object 2 for this arrangement is,

$$\dot{Q} = \left(\frac{1 - \varepsilon_1}{\varepsilon_1} + \frac{A_1 + A_2 - 2A_1F_{12}}{A_2 - A_1(F_{12})^2} + \left(\frac{1 - \varepsilon_2}{\varepsilon_2} \right) \frac{A_1}{A_2} \right)^{-1} A_1 \sigma (T_1^4 - T_2^4)$$

Heat Transfer: Glossary

Glossary

blackbody	A body with a surface emissivity of 1. Such a body will emit all of the thermal radiation it can (as described by theory), and will absorb 100% of the thermal radiation striking it. Most physical objects have surface emissivities less than 1 and hence do not have blackbody surface properties.
density, ρ	The amount of mass per unit volume. In heat transfer problems, the density works with the specific heat to determine how much energy a body can store per unit increase in temperature. Its units are kg/m ³ .
emissive power	The heat per unit time (and per unit area) emitted by an object. For a blackbody, this is given by the Stefan-Boltzmann relation σT^4
graybody	A body that emits only a fraction of the thermal energy emitted by an equivalent blackbody. By definition, a graybody has a surface emissivity less than 1, and a surface reflectivity greater than zero.
heat flux, q	The rate of heat flowing past a reference datum. Its units are W/m ² .
internal energy, e	A measure of the internal energy stored within a material per unit volume. For most heat transfer problems, this energy consists just of thermal energy. The amount of thermal energy stored in a body is manifested by its temperature.
radiation view factor, F_{12}	The fraction of thermal energy leaving the surface of object 1 and reaching the surface of object 2, determined entirely from geometrical considerations. Stated in other words, F_{12} is the fraction of object 2 visible from the surface of object 1, and ranges from zero to 1. This quantity is also known as the Radiation Shape Factor. Its units are dimensionless.
rate of heat generation, q_{gen}	A function of position that describes the rate of heat generation within a body. Typically, this new heat must be conducted to the body boundaries and removed via convection and/or radiation heat transfer. Its units are W/m ³ .
specific heat, c	A material property that indicates the amount of energy a body stores for each degree increase in temperature, on a per unit mass basis. Its units are J/kg-K.
Stefan-Boltzmann constant, σ	Constant of proportionality used in radiation heat transfer, whose value is 5.669×10^{-8} W/m ² -K ⁴ . For a blackbody, the heat flux emitted is given by the product of σ and the absolute temperature to the fourth power.

surface emissivity, ϵ

The relative emissive power of a body compared to that of an ideal blackbody. In other words, the fraction of thermal radiation emitted compared to the amount emitted if the body were a blackbody. By definition, a blackbody has a surface emissivity of 1. The emissivity is also equal to the absorption coefficient, or the fraction of any thermal energy incident on a body that is absorbed.

thermal conductivity, k

A material property that describes the rate at which heat flows within a body for a given temperature difference. Its units are W/m-k.

thermal diffusivity, α

A material property that describes the rate at which heat diffuses through a body. It is a function of the body's thermal conductivity and its specific heat. A high thermal conductivity will increase the body's thermal diffusivity, as heat will be able to conduct across the body quickly. Conversely, a high specific heat will lower the body's thermal diffusivity, since heat is preferentially stored as internal energy within the body instead of being conducted through it. Its units are m^2/s .