Theoretical Description and Numerical Sensitivity Analysis on Wilson Model for Hydraulic Transport of Solids in Pipelines.

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ABSTRACT

The Wilson model for the hydraulic transport of solids in pipelines is a widely used model. A theoretical background of the model has been published piece by piece in a number of articles over the years. A variety of information provided in these publications makes the model difficult to reconstruct.

A good understanding of the model structure is inevitable for the user who wants to extend or adapt the model to specific slurry flow conditions. An aim of this article is to summarise the model theory and submit the results of the numerical analysis carried out on the various model configurations. The numerical results show some differences when compared with the nomographs presented in the literature as the graphical presentations of the generalised model outputs. Model outputs are sensitive on a number of input parameters and on a model configuration used. A reconstruction of the nomographs from the computational model outputs is a subject to discussion.

Wilsonův model pro hydraulickou dopravu sypanin v potrubích je siroce používanou výhodou pomocí. Teoretický podklad modelu byl postupně v průběhu let zveřejněn v sérii publikací. Na základě téhoto publikace je vsak tezke zrekonstruovat model v jeho originální matematické podobě.

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INTRODUCTION

This article contains an overview of a theory for the Wilson two-layer model as it has been published in a number of articles over the years. Results are presented from the model computation. The results provide an insight to a behaviour of the mathematical model. The computation has been carried out using the MathCad document described by the authors in [2]. This MathCad document is available on a floppy disk at the authors upon request.

1. GEOMETRY OF TWO-LAYER MODEL

A schematic cross section of a pipe is illustrated in Fig.1 as it is defined in the two-layer model for the fully-stratified flow and for the heterogeneous (partially-stratified) flow.
The geometry of the pipe cross section is defined by the following equations.

The cross-sectional perimeters:

\[ L_1 = D(\pi - \beta) \]  
\[ L_2 = D\beta \]  
\[ L_{12} = D\sin\beta \]

The cross-sectional areas:

\[ A = \frac{1}{4}\pi D^2 \]  
\[ A_2 = \frac{1}{4}D^2(\beta - \sin\beta\cos\beta) \]  
\[ A_1 = A - A_2 \]

The equivalent hydraulic diameter of the non-circular waterway section above the bed is a function of the bed height [10]

\[ D_{eq} = \frac{4A_1}{L_1 + L_{12}} \]
2. FORCE BALANCE TO DETERMINE THE MDV CURVE AND THE RESISTANCE CURVE FOR FULLY-STRATIFIED FLOW

The Wilson model provides following parameters important for the slurry pipeline design and operation:

- the maximum deposit velocity \( V_{aMDV} \) (at the MDV curve). \( V_{aMDV} \) is the maximum average velocity of slurry flow in a pipe at which a stationary bed still occurs. The MDV curve depicts \( V_{aMDV} \) as a function of the bed height in a pipe.
- the friction loss (at the resistance curve). This curve depicts the pressure drop as a function of the flow rate in a pipe for slurry of the constant delivered concentration of solids.

Both curves can be plotted in one system of co-ordinates.

The MDV curve and the resistance curve are calculated from a force balance of four main forces (per unit length of the pipe) acting on the stationary or moving bed, which is formed by particles in mutual contact and contact with a pipe wall [4,5,6]. The force balance is written for forces and shear stresses averaged over the perimeters of flow boundaries.

The shear stresses on the flow boundaries are determined using Nikuradse friction equation for turbulent flow in a hydraulically-rough pipe [10]

\[
\tau = \frac{V^2 \rho}{\left(2.51 \ln \left( \frac{30 \text{D}_{eq}}{8r} \right) \right)^3}
\]  

(8)

The MathCad document solves a set of the model balance equations.

The equilibrium average velocity \( V_{eq} \) in the upper layer is obtained by solving \( V_1 \) in the force balance when the next quantities are kept constant:

- the bed height
- the bed velocity
- the physical properties of the fluid and solids.
V_{eq} is velocity V_1 for which the force balance is found by iteration in the MathCad document.

Following procedure is used for a model computation:

1. The **driving shear force on the bed surface** is calculated using the Nikuradse equation multiplied by an empirical constant for shear stress on the bed surface. This constant was originally assumed to be equal to the value 2 [6].

\[
F_{12} = 2L_{12}\tau_{12}
\]  

(9)

Shear stress \(\tau_{12}\) is calculated for velocity equal to the difference between velocity in the upper and in the lower layer.

2. The **driving force caused by the pressure gradient over a pipe section of a unit length** is determined from the pressure gradient

\[
\Delta P = \frac{F_{12} + \tau_{12}L_{12}}{A_1}
\]

(10)

and the driving force is

\[
F_2 = \Delta P \cdot A_2
\]

(11)

3. The resisting **mechanical friction force between bed and pipe wall** is determined; this is the normal force exerted by the bed against the pipe wall multiplied by the mechanical friction coefficient \(\mu\) [4,12].

\[
F_{2d} = g\rho\mu(S_x - S_f)C_b \frac{D^2}{2}(\sin\beta - \beta\cos\beta)
\]

(12)

4. The **viscous friction force between the bed and the pipe wall** is calculated

\[
F_{2v} = L_2\tau_2
\]

(13)

5. The **force balance** is

\[
F_{12}(V_{1,...}) + F_2(V_{1,...}) = F_{2d}(...) + F_{2v}(...)
\]

(14)

6. The **relative delivered concentration of solids** in slurry flow is determined as
Relative delivered concentration is a ratio of the absolute delivered concentration and concentration of solids in a loose-packed bed.

Wilson and his co-workers have published the nomographs [6,7,12] - the tools to predict the slurry flow parameters without handling the computational two-layer model. The nomographs are based on the computational model outputs. A comparison of the nomographic values with those from a computational model is of interest since it is not always clear for which slurry characteristics (as \(C_b\), \(\mu\), \(S_s\)) and model configuration the nomographs are proposed. The outputs of the computational model have been found very sensitive to the input parameters and a chosen model configuration.

**The resistance curve**

Any point of the resistance curve (\(i-V_a\) curve for const. \(C_{del}\)) is obtained by a numerical solution of the force-balance equations for the following conditions:

- constant bed velocity
- constant physical properties of the fluid and solids.

The bed height is a variable in a numerical iteration procedure. The bed height is determined for which two criteria are satisfied simultaneously:

- the force balance in pipe section is found
- the calculated delivered concentration equals the \(C_{del}\) required by the constructed resistance curve.

The resistance curve computed is presented in the same plot as a nomograph in the literature [12].

Following dimensionless parameters are handled in the nomograph:

- **Relative velocity** \(V_a/V_{max}\)
- **Relative concentration** \(C_{del}\)
the relative excess pressure gradient which is defined as

\[
\Delta P_{ex} = \frac{\Delta P - \Delta P_{clear}}{\Delta P_{plug}}
\]  

(16)

when \(\Delta P_{clear} = \frac{4 \tau}{D}\) is the pressure gradient of equivalent clear water flow and 
\(\Delta P_{plug} = 2\mu(S_s - S_f)C_s \rho\) is the pressure gradient for equivalent plug flow.

**The MDV curve**

Any point of the MDV curve is obtained by solving the force balance for a given bed height and \(V_2=0\). The curve is produced by solving the balance for an array of bed heights. A maximum at the MDV curve gives \(V_{max}\).

The MDV curve and the resistance curve are plotted in Fig 2. This figure is a product of the MathCad document described in [2].

![Figure 2: Non-dimensional MDV curve and resistance curve (fully-stratified flow).](image)

3. THE INCORPORATION OF SUSPENSION; HETEROGENEOUS MODEL

An adaptation of the two-layer model has been proposed [6] for the partially-stratified flow, i.e. flow in which a part of transported solid particles is suspended in the stream above the bed (see Fig. 1). Suspension of particles due to carrier turbulence causes an increase in the density (and viscosity at the highest concentrations) of mixture flow in
the upper layer [8]. This change in the physical properties of flow should explain a significant decrease of the $V_{\text{max}}$ with decreasing particle size (for particles smaller than approximately 0.7 mm) provided by the curve of the demi McDonald nomograph ($V_{\text{max}} = f(d, D, S_s)$) [7,12]. Although this decreasing trend can be produced by a numerical simulation of the model [2], it appears impossible to reproduce such a large drop in the $V_{\text{max}}$ values as the demi McDonald nomograph gives.

Wilson's (and his co-workers') investigation of the sheet flow has led to a further development in a structure of the two-layer model. Description of the flow in the shear layer, i.e. of the bed-load motion at high shear stress, has provided a new formulation of the friction law for an interface between bed and waterway.

A transition zone between packed granular bed and waterway above the bed is called shear layer. The model may be called 'three layer model' when the shear layer is implemented to its structure. At present the shear layer effect on the model structure is expressed only by an implementation of the new interfacial friction law to the two-layer model so not by changes in the model geometry.

4. THE THREE-LAYER MODEL

Publications [1,3,10,11,13,14] deal with a description of the shear on the bed-fluid interface.

Originally it was assumed [10] that the hydraulic roughness of the interface equals to one half of the shear layer thickness. The shear layer thickness is a function of the shear stress at the real/virtual interface. Thus shear stress was determined from a theoretical implicit equation in which the hydraulic diameter $D_{\text{eq}}$ was one of the variables.

Later Wilson & Nnadi [11] derived that the hydraulic diameter can be cancelled from the equations and that the friction factor at the bed surface depends only on $i/(S_s-1)$ providing the following relationship
\[ \frac{\delta_s}{R_b} = (C \tan \phi)^{-1} \left( \frac{i}{S_s - 1} \right) \]  

(17)

\( R_b \) should be determined using a method from [3]. An application of the eq. (7) has led to the following semi-empirical formula expressing a friction law for sheet flow [11]

\[ f_{12} = 0.088 \left( \frac{i}{S - 1} \right)^{0.22} \]  

(18)

revised in [13] as

\[ f_{12} = 0.87 \left( \frac{i}{S - 1} \right)^{0.78} \]  

(19)

Eq.(17) has been also implemented to the general friction equation for rough-wall boundary

\[ \frac{V}{\sqrt{\tau_{12}}} = \sqrt{8} \left( \frac{f_{12}}{f_{12}} \right) = \frac{1}{\kappa} \ln \left( \frac{R_b}{\delta_s} \right) + B \]  

(20)

Empirical constants in the eq. (20) have been determined by a calibration of the eq. (20) by the experimental data. Different constants have been published for different data (characterised here by different \( \phi \)):

equation published in [1](for \( \phi = 240^\circ \))

\[ \frac{V}{\sqrt{\tau_{12}}} = 2.7 - 2.5 \ln \left( \frac{i}{S - 1} \right) \]  

(21)

equation published in [14](for \( \phi = 180^\circ \))

\[ \frac{V}{\sqrt{\tau_{12}}} = 2.5 \ln \left( 2.2 \left( \frac{S - 1}{i} \right) \right) \]  

(22)

The equations (19, 21, 22) give similar \( f_{12} \) values but the eq. (18) differs.

When the recently published value \( \phi = 14^\circ \) [13] for a tested material is used in the eq. (20) the following equation can be written
\[
\frac{V}{\tau_{12}} = 1.2 - 2.5\ln\left(\frac{i}{S_{s} - 1}\right)
\]  

(23)

Recently, Wilson has proposed a correction of the demi McDonald nomograph based on analytical results from the three-layer model. This has a form of a fit function [12,13]. The three-layer model outputs have shown that \( V_{\text{max}} \) has not been dependent on the particle diameter when the friction law for sheet flow has been used for the interface between layers

\[
V_{\text{max}} = \sqrt{2gD(S_{s} - 1)\left(\frac{0.018}{f_{f}}\right)^{0.13}}
\]  

(24)

Wilson & Pugh [13] have recommended to use this equation instead of the curve in the demi McDonald nomograph when the value of \( V_{\text{max}} \) obtained from the demi McDonald nomograph exceeds that from the fit function.

The three-layer model has been tested in the MathCad document [2]. The \( V_{\text{max}} \) outputs for various friction equations are compared with the fit function in Fig. 3. Following input parameters to the model are used: \( \mu=0.4, \tau=10^{-5}, C_{b}=0.6, S_{s}=2.65 \) and \( f_{f} \) according to Nikuradse.

![Figure 3: Maximum deposit velocity. Comparison of the fit function with the outputs of the three-layer model for various interface-friction equations.](image)

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The fit function (24) matches reasonably the three-layer model outputs for all tested friction equations. The best fit is reached by the eq. (22). A decrease in $\phi$ from 29 to 14 degrees causes a decrease in $V_{\text{max}}$ of 15 - 20%.

5. DISCUSSION AND CONCLUSIONS

The theoretical background of the Wilson model for fully-stratified flow, heterogeneous flow and stratified flow with a shear layer has been examined. Model configurations can numerically analysed in the MathCad document. Examples of the analysis are presented on Fig. 2 and Fig. 3. Issues from an extensive testing are generalised to the following remarks regarding a configuration and an application of the computational model and the nomographs.

1. The viscous bed-wall friction and horizontal asymptote of resistance curves

It was assumed originally that viscous friction between bed and pipe wall was that for clear water at the pipe wall for the average velocity equalled to velocity of the sliding bed [6]. Then the graph given in [12] can be reproduced by the outputs of the two-layer model as shown in Fig. 2.

Wilson & Brown have later published [9] a method for a determination of viscous friction between sliding granular bed and pipe wall. They compared the viscous friction between a sliding bed and a pipe wall to the friction between a capsule and a pipe wall. According to their analysis the **viscous friction factor** and **wall shear stress** should be determined according to the following procedure.
If \( \text{Re}_2 = \frac{V_2 \rho}{\eta} < 335 \) then

\[
f_2 = \frac{22}{\text{Re}_2}
\]

If \( \text{Re}_2 > 335 \) then

\[
f_2 = 0.033 \left(1 + \frac{138}{\text{Re}_2}\right)^2
\]

The shear stress is

\[
\tau_2 = \frac{f_2 \rho V_2^2}{8}
\]

When this method is implemented to the computational model, the resistance curve no longer has a horizontal asymptote as shown in Fig. 4.

Figure 4: Non-dimensional MDV curve and resistance curve from the model with implemented viscous friction \( f_2 \) according to [9] (fully-stratified flow).

Thus an implementation of this method is not appropriate for the two-layer model. An absence of the horizontal asymptote in Fig. 4 can be explained from the following. The proposed method provides higher viscous shear stress between bed and pipe wall than is that for fluid. Therefore ratio \( V_{\text{eq}}/V_2 \) increases with increasing \( V_a \) when the slurry...
flow is simulated for a given bed height. This results in a decrease in the delivered concentration because all solids are delivered by the lower layer according to the model structure. To maintain a constant delivered concentration (as required by a resistance curve of constant $C_{\text{del}}$), the bed height must increase with increasing $V_a$. Thicker granular bed provides more resistance and so higher pressure gradient in a pipe.

2. The cross section between the MDV curve and the resistance curve - zero delivered concentration at the MDV curve

A determination of the MDV curve and the resistance curve in the plot $\Delta p_{\text{ex}}$ vs. $V_a/V_{\text{max}}$ (Fig. 2, 4) is based on the fully-stratified flow pattern. It is assumed that no particles are delivered until the average velocity in a pipe prevail the critical value determined by the MDV curve. In most real flow situations some portion of solids is delivered also at the average velocities below the critical value for which granular bed starts to slide. This is caused by a suspension of particles due to high fluid velocity in the upper layer and/or by a development of a shear layer at the top of a granular bed. Therefore the resistance curves for the low delivered concentrations should cross the MDV curve.

3. Empirical constant for a determination of the friction factor at the layers interface

Numerical simulations have shown that the multiplication coefficient proposed for the Nikuradse equation to determine the interfacial friction factor does not reproduce the demi McDonald curve. The coefficient equal to 2.75 (instead of 2.00) provides model outputs matching the demi McDonald curve for particle sizes for which the fully-stratified flow is expected (approx. $d > 0.7$ mm). Even higher value of the coefficient would have to be used to reproduce the demi McDonald curve for heterogeneous flow (a curve section for approx. $d < 0.7$ mm).
NOMENCLATURE

A  cross-sectional area of pipe
A₁  cross-sectional area of upper layer
A₂  cross-sectional area of lower layer
B  empirical coefficient
C  volumetric concentration of solids in shear layer
Cₜb  volumetric concentration of solids in the loose-packed bed
C_{del}  relative delivered concentration of solids
d  particle diameter
D  pipe diameter
D_{eq}  equivalent hydraulic diameter
f  Darcy-Weisbach friction factor for fluid flow
f₁₂  Darcy-Weisbach friction factor at stratified-flow interface
f₂  Darcy-Weisbach friction factor for bed flow
F₁₂  driving force on the surface of contact layer
F₂  driving force to contact layer due to pressure gradient
F₂d  mechanical friction force of contact layer against pipe wall
F₂v  viscous friction force between lower layer and pipe wall
g  gravitational acceleration
i  hydraulic gradient
L₁  perimeter of pipe between upper layer and pipe wall
L₁₂  perimeter of interface between upper layer and lower layer
L₂  perimeter of pipe between lower layer and pipe wall
ΔP  pressure gradient for mixture flow
ΔP_{clear}  pressure gradient for clear water flow
ΔP_{ex}  relative excess pressure gradient
ΔP_{plug}  pressure gradient for plug flow
r  absolute roughness of flow boundary


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R_{b} \quad \text{hydraulic radius associated with bed}

Re_{2} \quad \text{Reynolds number}

S_{f} \quad \text{relative density of fluid}

S_{s} \quad \text{relative density of solids}

V \quad \text{average velocity in waterway}

V_{a} \quad \text{average slurry velocity in full cross-sectional area of pipe}

V_{aMDV} \quad \text{value of } V_{a} \text{ at limit of deposition}

V_{eq} \quad \text{average velocity in upper layer for which force balance is found}

V_{\text{max}} \quad \text{maximum value of } V_{aMDV}

V_{1} \quad \text{average velocity in upper layer}

V_{2} \quad \text{average velocity in lower layer}

\alpha \quad \text{angle defining position of surface of real/virtual interface}

\beta \quad \text{angle defining position of surface of contact-load layer}

\delta_{s} \quad \text{thickness of the shear layer}

\eta \quad \text{dynamic viscosity of fluid}

\kappa \quad \text{von Karman constant}

\mu \quad \text{mechanic friction coefficient of solids against pipe wall}

\rho \quad \text{density of fluid}

\tau \quad \text{shear stress at waterway boundary}

\tau_{1} \quad \text{shear stress between upper layer and pipe wall}

\tau_{2} \quad \text{shear stress between granular bed and pipe wall}

\tau_{12} \quad \text{shear stress at stratified-flow interface}

\phi \quad \text{angle of internal friction of particles (dynamic)}

\text{Abbreviation:}

MDV \quad \text{maximum deposit velocity}
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