Chapter 1

INTRODUCTION

The development of this text book has been driven by the needs of engineers working in the sea. This subsection lists a number of questions which can result from very practical general offshore engineering situations. It is the intention that the reader will become equipped with the knowledge needed to attack these problems during his or her study of this book; problems such as:

- How do I determine the design (hydrodynamic) loads on a fixed offshore tower structure?
- Can a specified object be safely loaded and carried by a heavy lift float-on float-off vessel?
- What is the optimum speed of a given supply boat under the given sea conditions?
- How should a semi-submersible platform be ballasted to survive an extreme storm?
- Under what conditions must a floating drilling rig stop work as a result too much motion?
- How important is the heading of a drilling ship for its behavior in waves?
- What dynamic positioning power is needed to keep a given drilling ship on station under a given storm condition?
- How will the productivity of a marine suction dredge decline as the sea becomes rougher?
- What sea conditions make it irresponsible to transfer cargo from a supply boat to a fixed or another floating platform?
- How does one compute the wave and current forces on a large truss-type tower structure in the sea?
- How can the maximum wave and current loads on a truss-type tower structure be estimated efficiently?

• What sea bed changes can be expected near a pipeline or small subsea structure?

In contrast to some other books, this one attempts to prevent a gap from occurring between material covered here and material which would logically be presented in following texts. For example, after the forces and motions of a ship have been determined in chapter 8, the treatment continues to determine the internal loads within the ship. This forms a good link to ship structures which will then work this out even further to yield local stresses, etc.

1.1 Definition of Motions

![Figure 1.1: Definition of Ship Motions in Six Degrees of Freedom](image)

The six ship motions in the steadily translating system are defined by:

• three translations of the ship’s center of gravity (CoG or G) in the direction of the x-, y- and z-axes:
  - surge in the longitudinal x-direction, positive forwards,
  - sway in the lateral y-direction, positive to port side, and
  - heave in the vertical z-direction, positive upwards.

• three rotations about these axes:
  - roll about the x-axis, positive right turning,
  - pitch about the y-axis, positive right turning, and
  - yaw about the z-axis, positive right turning.

These definitions have been visualised in figure 1.1.
Any ship motion is build up from these basic motions. For instance, the vertical motion of a bridge wing is mainly build up by heave, pitch and roll motions. Another important motion is the vertical relative motion, defined as the vertical wave elevation minus the local vertical motion of the ship. This is the motion that one observe when looking over the rail downwards to the waves.

1.2 Problems of Interest

This section gives a brief overview of fixed and mobile offshore units, such as dredgers, pipe laying vessels, drilling vessels, oil production, storage and off-loading units and several types of support and transportation vessels. Their aspects of importance or interest with respect to the hydromechanical demands are discussed. For a more detailed description of offshore structure problems reference is given to a particular Lecture on Ocean Engineering by [Wichers, 1992]. Some relevant knowledge of that lecture has been used in this section too.

1.2.1 Suction Dredgers

Dredging is displacement of soil, carried out under water. The earliest known dredging activities by floating equipment took place in the 14th century in the harbor of the Dutch Hanze city of Kampen. A bucket-type dredging barge was used there to remove the increasing sand deposits of the rivers Rhine and IJssel. Generally, this work is carried out nowadays by cutter suction dredgers or trailing suction hopper dredgers. The cutter suction dredger is moored by means of a spud pile or mooring lines at the stern and by the ladder swing wires at the bow. These dredgers are often used to dredge trenches for pipe lines and approach channels to harbors and terminals in hard soil. The trailing suction hopper dredger is dynamically positioned; the dredger uses its propulsion equipment to proceed over the track.

The environmental sea and weather conditions determine:

- the available working time in view of:
  - the necessity to keep the digging tools in contact with the bottom, such as dippers, grabs, cutters, suction pipes and trail heads,
  - the anchorage problems in bad weather, such as breaking adrift from anchors and bending or breaking of spud piles,
  - the stability of the discharge equipment, such as floating pipelines and conveyor belts,
  - the mooring and stability of barges alongside in the event of currents and/or high wind velocities and waves and
  - the overloading of structural elements associated with dredging such as bucket-ladders or cutter arms,

- maneuverability, especially at strong side winds and strong currents entering at a specific angle, which is important for the dredging slopes,

- problems on slamming of bottom doors of sea-going hopper barges and on jumping suction pipes of hopper suction dredgers and

- hopper overflow losses due to excessive rolling of the vessel.
Cutter suction dredgers are moored by means of a spud pile or radially spread steel wires. An important feature of spud pile mooring (see figure 1.2) is the relative high stiffness, compared to other mooring systems. One of the reasons for this high stiffness is the required accurate positioning of the cutter head in the breach. This is necessary for a good dredging efficiency. Another reason is to avoid to high loads on the cutter head and on the ladder. Since the spud pile can only take a limited load, the workability limit in waves is at significant wave heights between 0.5 and 1.0 m depending on the size of the dredger, the wave direction and the water depth.

Figure 1.2: Cutter Suction Dredger with Fixed Spud Piles

Cutter suction dredgers may also be equipped with a softer mooring system. In this case the spud pile is replaced by three radially spread steel wires attached to the spud keeper. This so-called ‘Christmas tree’ mooring results in lower loads but larger wave induced motions of the dredger. Thus, larger workability has to be traded off against lower cutting efficiency. Other components of the dredging equipment - such as the floating pipe line to transport the slurry, its connection to the dredger and the loads in the ladder - may also be limiting factors to the workability of the dredger in waves.

Aspects of importance or interest of cutter suction dredgers are:
- a realistic mathematical modelling of the soil characteristics for simulations,
- the loads in the spud pile, the cutter head and the ladder hoist wires,
- the motions of the dredger and the cutter head in the breach and
- the loads in the swing wires.

Trailing suction hopper dredgers are used for maintenance work (removal of deposits in approach channels) and dredging of trenches or approach channels in softer soils. It is a ship-shape vessel with hopper type cargo holds to store the slurry. At each side of the ship is a suction arm, which consists of a lower and a higher part, connected through cardanic joints. The connection to the ship is through upper joints and stringers. On modern suction dredgers, the support wire connected to the lower part of the suction pipe near the suction head is provided with a constant tension device for compensation of heave motions. Figure 1.3 shows an example of the increased size of these vessels between 1962 and 1997. This type of dredgers sail along a track during the dredging operation. For manoeuvering, the main propeller(s), the rudder(s) and a combination with bow and/or stern thrusters are used. For that purpose the helmsman manually controls the vessel by using a monitor.
1.2. PROBLEMS OF INTEREST

1-5

Figure 1.3: Trailing Cutter Suction Hoppers

showing the actual position of the master drag head and the desired track. Operating in near-shore areas, the vessel will be exposed to waves, wind and current and manual control of the vessel may be difficult. An option is an automatic tracking by means of DP systems. Aspects of importance or interest of trailing suction hopper dredgers are:

- the motions of the vessel in waves,
- the current forces and the wave drift forces on the vessel,
- the low speed tracking capability,
- the motions of the suction arms,
- the loads in the suction arm connections to the vessel,
- the effect and interactions of the thrusters and
- the avoidance of backward movements of the suction head at the sea bed.

1.2.2 Pipe Laying Vessels

One of the problems of laying pipes on the sea bed lies in the limited capacity of the pipe to accept bending stresses. As a result, it is necessary to keep a pipe line under considerable axial tension during the laying operation otherwise the weight of the span of the pipe between the vessel and the point of contact of the pipe on the sea bed will cause the pipe to buckle and collapse. The tension in the pipe line is maintained using the anchor lines of the pipe laying vessel. The tension is transferred to the pipe by means of the pipe line tensioner located near the point where the pipe line leaves the vessel. The pipe tensioner is designed to grip the pipe without damaging the pipe coating (concrete) and ease the pipe aft while retaining pipe tension as the vessel is hauled forward by the anchor winches. Forward of the tensioner, additional sections of the pipe line are welded to the already existing part. Aft of the tensioner, part of the free span of the pipe line is supported by a so-called stinger.
CHAPTER 1. INTRODUCTION

Pipe laying vessels can consist of semi-submersibles or ship-shaped hulls. Semi-submersibles have the advantage of better motion characteristics in waves which is beneficial for the pipe laying operation. On the other hand, ship-shaped vessels have a higher variable load capacity and a much higher transit speed.

Pipe laying vessels are usually moored by means of anchor systems which are continually being relocated as the laying operation progresses. A new development is a pipe laying vessels kept in position and deriving the pipe line tension by using a dynamic positioning (DP) system instead of anchor lines; for instance pipe laying vessel 'Solitaire', operated since 1998 by Allseas Marine Contractors. Figure 1.4 shows a comparison of this vessel with the much smaller 'Lorelay' of this company, operated since 1986. The considerable increase of size is obvious here.

![Figure 1.4: Pipe Laying Vessels 'Lorelay' (1986) and 'Solitaire' (1998)](Source: Wichers, 1992)

With respect to the combined effect of vessel and pipe motions and the dynamic positioning performance, analyses have to be carried out to find the sea state that can be regarded as the maximum operational condition. In order to determine the most critical wave direction, these analyses have to be run for a number of wave directions. As an indication for this sea state can be found: \[ H_{1/3}^{\text{max}} = 3.0 \text{ m} \] for the 150 m length pipe laying vessel 'Lorelay' and \[ H_{1/3}^{\text{max}} = 4.0 \text{ m} \] for the 250 m vessel 'Solitaire'. In many cases, bow-quartering environmental conditions will be found as the most critical operational condition with regard to vessel motions, pipe movements at the stinger and the DP performance. In order to establish whether the DP system of a pipe laying vessel is still redundant or close to its maximum capability as a result of forces exerted by the environment and the pipe tension, a DP analysis should be performed which forms an integral part of the dynamic analyses. The results of these analyses have to be used to determine nominal and maximum stinger tip clearance values. This information is required to assist offshore personnel in determining nominal lay tension during pipe laying operations with changing environmental conditions. If moored by means of anchor systems, the ability to lay pipes in severe sea conditions is determined by:
- the stiffness of the mooring system,
- the forces in the mooring lines,
- the holding capacity of the anchors,
- the possibility for anchor handling,
- the wave frequency motions of the vessel,
- the low frequency horizontal (surge, sway and yaw) motions of the vessel,
- the forces exerted by the stinger on the stinger-vessel connections and
1.2. PROBLEMS OF INTEREST

- the buckling and bending stresses in the pipe line.

In case of dynamic positioning, the accuracy of the low speed tracking capability is an important aspect too.

After laying the pipe, it has - in many cases - to be buried in the sea bed or to be covered by gravel stone. Another possibility is to tow a trencher along the pipe, which acts as a huge plow. The trencher lifts the pipe, plows a trench and lowers the pipe trench behind it. The sea current takes care of filling the trench to cover the pipe, as will be discussed in chapter 14.

In very extreme conditions the pipe is plugged and lowered to the sea bed, still keeping sufficient tension in the pipe in order to avoid buckling. After the pipe is abandoned, the vessel rides out the storm. The ship has to survive a pre-defined extreme sea state, for instance a 100-years storm in the North Sea.

1.2.3 Drilling Vessels

When geological predictions based on seismic surveys have been established that a particular offshore area offers promising prospects for finding oil, a well is drilled to examine these predictions. These drilling operations are carried out from barges, ships, semi-submersibles or jack-up rigs; see figure 1.5.

![Drilling Vessels Diagram](image)

Figure 1.5: Ship and Semi-Submersible Type of Drilling Vessel

For floating drilling units it is important that they can work in high sea conditions. Generally, semi-submersibles will have the least vertical motions and hence the highest workability. In order to enhance further the workability of drilling vessels, the vertical motions of the vessel at the location of the drill string are compensated for by a so-called heave...
compensator. This device is essentially a soft spring by means of which the drill string is suspended from the drilling tower on the vessel. In this way, the drill string can be maintained under the required average tension, while the vertical motions of the vessel - and hence of the suspension point of the drill string in the drilling tower - do not result in unduly high drill string tension variations or vertical movement of the drill bit. Also the marine riser is kept under constant tension by means of a riser tensioning system.

Horizontal motions are another factor of importance for drilling vessels. As a rule of thumb, the upper end of the drill string may not move more to one side of the center of the drill than approximately 5% of the water depth. This places considerable demands on the mooring system of the vessels. Semi-submersible drilling vessels are generally moored by means of 8 to 12 catenary anchor legs. Drill ships can be moored by means of a spread mooring system which allows a limited relocation of the vessel’s heading or by means of a dynamic positioning (DP) system. The latter system of mooring is also used for semi-submersible drilling vessels for drilling in water exceeding 200 m depth.

Summarized, aspects of importance or interest of drilling vessels are:
- the wave frequency vertical motions at the drill floor,
- the horizontal motions at the drill floor,
- the deck clearance of the work deck of a semi-submersible above a wave crest (air gap),
- the vertical relative motions of the water in the moonpool of drill ships,
- the damage stability of a drilling semi-submersible,
- the forces in the mooring lines,
- the mean and low frequency horizontal environmental forces on DP ships and
- the thruster effectiveness in high waves and currents.

As an example, some limiting criteria for the ship motions - used for the design of the 137 m length drilling ship 'Pelican' in 1972 - were:
- ship has to sustain wind gusts up to 100 km/hour,
- maximum heel angle: 3 degrees,
- roll: 10 degrees in 10 seconds,
- pitch: 4 degrees in 10 seconds and
- heave: 3.6 meter in 8 seconds.

**1.2.4 Oil Production and Storage Units**

Crude oil is piped from the wells on the sea bed to the production platform where it is separated into oil, gas, water and sand. After purification, the water and sand returns to the sea. Gas can be used for energy on the platform, flared away or brought to shore by means of a pipe line. In some cases the produced gas or water is re-injected in the reservoir, in order to enhance the production of crude oil. Crude oil is piped to shore or pumped temporarily into storage facilities on or near the platform.

**Jackets**

Fixed platforms for oil and gas production are used at water depths ranging to about 150 m. In most cases they consist of a jacket, a steel frame construction piled to the sea bed; see figure 1.6. The jacket supports a sub-frame with production equipment and accommodation deck on top of it.

The jacket has to be transported on a barge to its installation site at sea. During transport, the aspects of importance or interest of a jacket are:
1.2. PROBLEMS OF INTEREST

- the accelerations and motions of the jacket,
- the forces exerted by the sea-fastenings on the jacket and
- the wave impacts on overhanging parts of the structure.

During the installation phase of the jacket, it has to be launched from its transportation barge; see figure 1.7. Then, the jacket is temporarily a floating structure.

Figure 1.7: Schematic View of Launching a Jacket from a Barge

During launching, the aspects of importance or interest of the jacket are:
- the motions of the jacket on the transportation barge in waves,
- the forces exerted on the jacket by the launchways and rocker arm beams,
- the maximum depth to which the jacket dives and
- the static as well as the dynamic stability of the jacket.

Then, the free-floating jacket has to be up-ended before it can be put on the sea bed. During this installation phase, the aspects of importance or interest of a jacket are:
- the flooding sequence of the ballast tanks,
- the hydrostatic stability of the jacket,
- the current forces on the jacket and
- the internal stresses in the jacket.

When the jacket is ballasted to stand on the sea bed, in some cases it will have to be positioned over the pre-drilled template; see figure 1.6. The guiding pins have to enter the receiver cans on the template. In many cases this operation is carried out with help of a crane vessel. During this installation phase, the aspects of importance or interest of a jacket are:
- the motions in waves,
- the current forces,
- the internal stresses in the jacket,
- the loads in the guiding pins and
- hoist wave loads in case of crane assisted positioning.

Finally, the jacket will be anchored to the sea bed by means of long steel pipes which are lowered through the pile sleeves and hammered into the sea bed. This can be done by a crane vessel, which also installs the top-side modules.

Jack-Ups

A jack-up is a mobile drilling unit that consists of a self-floating, flat box-type deck structure supporting the drilling rig, drilling equipment and accommodation; see figure 1.8. It stands on 3 or 4 vertical legs along which the platform can be self-elevated out of the water to a sufficient height to remain clear of the highest waves. Drilling operations take place in the elevated condition with the platform standing on the sea bed. This type of platform is used for drilling operations in water depths up to about 100 m. Jack-ups spend part of their life as floating structures. This is when such platforms are towed to a new location by means of ocean-going tugs. In this mode, the legs are lifted up and extend upwards over the platform.

Aspects of importance or interest in the towing condition of a jack-up are:
- the towing resistance,
- the course stability under tow and
- the bending stresses in the legs at the point of connection to the deck structure.

These bending stresses are adversely affected by the wave induced roll and pitch motions of the vessel. In some cases the highest sections of the legs are removed for the towing operation in order to reduce the bending stresses to acceptable levels.
1.2. PROBLEMS OF INTEREST

Gravity Base Structures

Gravity Base Structures (GBS) are applied to remote fields in deep and harsh waters in the central and northern part of the North Sea. They consist of a combination of a number of large diameter towers, placed on top of a large area base which contains also storage capacity. Piling to the sea bed is not required because of the large size of the base and the mass of the structure, but the sea bed has to be leveled. The towers support a sub-frame with a production equipment and accommodation deck on top of it. Figure 1.9 shows some examples of large gravity base structures.

After construction inshore, the unit is mated with the top-side structure in relatively sheltered waters. Aspects of importance or interest during this mating phase of a GBS are:
- the positioning of the top-side relative to the main structure and
- the impact loads between the top-side structure and the main platform structure.

Both these aspects require very mild environmental conditions during the actual mating. Then the complete platform is towed out to the work location by 4 to 6 tugs. Aspects of importance or interest during this towing phase of a GBS are:
- the towing force,
- the course stability under tow,
- the 'keel' clearance at restricted water depths, influenced by vertical platform motions,
- the wave induced motions,
- the motions due to vortex shedding and flow separation around slender columns and
- the current loads.

At the work location, the platform is being ballasted down to the sea bed. Aspects of importance or interest during this ballasting down phase of a GBS are:
- the positioning accuracy,
- the vertical motions on setting down and
- the horizontal wave, wind and current forces.

Floating Production Units

Floating Production units (FP) are used for oil production at smaller fields. In very deep water they are also economically attractive for large oil fields. The production equipment and accommodation is placed on a floating structure, permanently moored to the sea bed. They have no storage capacity. The oil is off-loaded to a Floating Storage and Off-loading unit (FSO). Figure 1.10 shows an overview of this.

![Floating Production Unit with External Storage Facilities](source)

Two types of floating production platforms without storage capacity can be distinguished: the semi-submersible and the tension leg platform (TLP). Production on a TLP in remote areas - so that a pipe line is uneconomical - requires external storage capacity, for instance by using a storage tanker moored nearby.

A Semi-Submersible Platform consists of a rectangular deck structure supported by 4 to 8 surface-piercing vertical columns standing on submerged horizontal floaters. These vessels have good motion characteristics and do not require the heading changed as the predominant direction of the weather changes. The vessels are moored by means of 8 to 12 catenary mooring lines consisting of chains or combinations of chain and wire. Parts of the pipe lines transporting the oil to the floater have to be flexible to allow for the wave induced motions of the floater. These flexible pipe lines have to be sufficiently strong and resilient to withstand high pressures and temperatures of the crude oil as well as the continual flexing due to the floater motions; see figure 1.11.

The aspects of importance or interest are generally the same as those for drilling semi’s. However, the production semi-submersible will be permanently moored which means that - consequently - more stringent demands are placed on the design. Additional aspects of importance or interest of semi-submersible production platforms are:

- the behavior of bundles of flexible flow lines and risers and
1.2. PROBLEMS OF INTEREST

A Tension Leg Platform (TLP) consists of a semi-submersible type hull with for instance four vertical surface-piercing columns standing on underwater floaters and supporting a large rectangular deck; see figure 1.12. At each of the four corners of the floater, pre-tensioned tethers extend vertically downwards to foundation templates which are piled into the sea bed. Due to the vertical tendons, which are pre-tensioned to such a degree that they never become slack, any vertical motion of the TLP will be eliminated. This allows for steel pipe line connections between the wells and the floater, without the need for flexible sections of pipe lines. As a result, it is possible to install the well head control valves on the deck of the floater instead of on the sea bed. This represents a considerable advantage from the point of view of ease of maintenance and investment.

The installation phases consist of construction of the floaters in a dry dock inshore, float out, deck mating, tow out to the production field and the final installation. Aspects of importance or interest during the installation phase of a TLP are:
- the avoidance of vessel impact loads during the deck mating operation,
- the horizontal positioning accuracy during deck mating and hook-up of the tethers and
- the towing resistance.

Aspects of importance or interest in the installed condition of a TLP are:
- the avoidance of wave impacts on the under side of the deck,
- the wave and low frequency horizontal forces and motions,
- the wave and current induced vortices,
- the avoidance of slack tethers,
- the maximum tether tensions,
- the behavior of the bundles of the tethers,
- the super-harmonics in tether tensions (springing) and
- the current and wind forces.

**Floating Storage and Off-loading Units**

Permanently moored Floating Storage and Off-loading vessels (FSO’s) are used to store the produced crude oil. Periodically, the oil is collected and transported to shore by means of shuttle tankers. For the off-loading operation, the shuttle tanker is often moored either in tandem with the storage vessel or alongside. Sometimes the stored oil is piped to a Single Point Mooring system (SPM) some distance away, to which the shuttle tanker is temporarily moored. A Spar buoy mooring system is an example of this combined offshore storage and mooring facility; see figure 1.13.

Generally, due to costs aspects, existing tankers - with sizes ranging from 80 to 200 kDWT - have been used as storage vessels, permanently moored in the neighborhood of a production platform or a group of platforms. When converting oil tankers for this purpose, special attention has to be paid to the remaining fatigue life of this older vessel. Nowadays, the number of suitable tankers on the market is relatively small and a trend toward purpose built storage vessels is discernible.

A factor of prime importance for the operation of these vessels is the continued integrity of the mooring system and of the pipe line carrying the crude to the storage vessel. Another important aspect is the operational limit of the crude oil transfer operation between the storage vessel and the shuttle tanker. Both these design requirements are mainly determined by the wind, wave and current induced motions, the mooring forces of the storage vessel and those of the shuttle tanker.

Aspects of importance or interest of an FSO are:

![Tension Leg Platform](source: internet)
1.2. PROBLEMS OF INTEREST

- design loads of the mooring system for extreme sea conditions (100-year storm),
- the environmental loads on and fatigue aspects of mooring system components,
- the loads in the shuttle-to-storage vessel mooring lines,
- the mean and low frequency environmental forces on the storage vessel,
- the operational limits with respect to shuttle tanker loading,
- the approach manoeuvre of the shuttle tanker,
- the installation aspects of the storage vessel mooring and
- the behavior of (flexible) flowlines in extreme conditions.

Floating Production, Storage and Off-loading Vessel

A Floating Production, Storage and Off-loading vessel (FPSO) is generally based on the use of a tanker hull, which has been converted for the purpose. Such vessels have a large storage capacity and deck area to accommodate the production equipment and accommodation; see figure 1.14. When converting old tankers for this purpose, special attention has to be paid to the fatigue life of the vessel.

Motion characteristics of such vessels are acceptable as long as the vessel can 'weathervane' with the predominant direction of the wind and the waves. This requires that a single point mooring system be used by means of which the vessel is effectively held at the bow or stern by the mooring system and allowed to rotate freely around that point. A complicating factor of the SPM system is the need to include fluid swivel systems in the oil transport system to and from the vessel.

In some sheltered locations it is not necessary to apply an SPM type mooring system. In such cases a spread mooring system which holds the vessel in a fixed mean heading direction is the preferred solution since no swivels are required in the oil transport lines to and from the vessel. Due to the wave induced motions of the FPSO, the oil transportation lines to the vessel have to be flexible.
Aspects of importance or interest are mainly similar to those for an FSO system, discussed before. However, two additional aspects for FPSO’s are:
- the effect of the vessel motions on the production process and
- the vertical relative motions between the waves and the deck edges (deck wetness).

1.2.5 Support Vessels

Some requirements on support vessels - such as seismic and survey vessels, stand-by vessels, anchor handling vessels, well maintenance vessels, diving support vessels and crane vessels - are briefly described here.

Seismic and Survey Vessels

The decision to drill at a given location is based on the result of seismic and geological surveys of the structure underlying the sea bed. Some seismic survey systems are shown in figure 1.15.

Whereas in earlier days of the offshore industry existing utility vessels were used as a base for carrying out such surveys, nowadays purpose built vessels extensively equipped with sophisticated seismic data gathering and analysis systems are appearing on the market. The success rate of finding oil and gas fields has increased considerably since the introduction in recent years of three-dimensional seismic survey method. Aspects of importance or interest of this type of vessels are:
- the wave frequency motions and accelerations,
- the vertical relative motions between the waves and the deck edges (deck wetness),
- the low speed tracking capability,
- the station keeping ability and
- the workability of the specialists onboard.
1.2. PROBLEMS OF INTEREST

Well Maintenance Vessels

Throughout the productive life of an oil or gas well, maintenance of the well plays an important role. Well maintenance generally requires that a so-called 'work-over unit' be brought overhead of the wellhead. The well is entered and tools are lowered into the well. In the past, use was generally made of drilling vessels since these were fitted with the necessary equipment for carrying out the positioning operation, the entering of the well and the lowering of the tools into the well. In recent years, however, there is a trend to using smaller, more cost effective vessels for this work. These vessels, while possessing the required equipment in the form of a drilling tower etc., do not need to have other facilities which are specifically needed for drilling operations, such as a large load carrying capacity for the purpose of the storage of large amounts of drill string or well casing or for drilling fluids. Maintenance vessels can therefore be smaller and less expensive. At present a number of smaller semi-submersibles are in use for this purpose. Well maintenance has also been carried out from the decks of larger supply vessels which have been fitted with the necessary work-over equipment.

Aspects of importance or interest are the same as mentioned for the semi-submersible drilling rigs. An increasing number of this type of vessels is being fitted with DP systems. This type of mooring system is very flexible and avoids the complications involved in deploying anchor lines over a sea bed that may have several pipe lines situated on it.

Diving Support Vessels

Diving support vessels are intended as a base for diving operations offshore. Many of the subsea activities require the presence of divers subsea, as the development of robot techniques has not yet progressed far enough to be applicable to all activities. The offshore industry moves into deeper waters in its search for oil and gas, so diving operations must be carried out in ever deeper water. Safe diving operations in deep water require even more sophisticated support systems, such as large pressure vessels and diving bells to allow the divers to remain under deep water pressure while on board. This makes deep water diving practical, since it is not necessary to depressurize in between working spells. Diving support vessels are fitted with all equipment necessary for safe deep water diving operations,
CHAPTER 1. INTRODUCTION

involving a number of divers simultaneously. Besides sophisticated diving equipment, such vessels are often equipped with DP systems. Since the divers operating on the sea bed depend on the vessel, high demands are placed on the integrity of the DP systems and on the motion characteristics of such vessels in waves.

Aspects of importance or interest of diving support vessels are:
- the horizontal wind, wave and current loads,
- the wave frequency motions and accelerations,
- the vertical relative motions in the moonpool,
- the effect of current and wave frequency motions on the thruster performances and
- the station keeping ability.

Crane Vessels

Offshore construction - such as involved in building a fixed offshore production platform - requires a large number of crane lifts in which the construction elements (so-called modules) are lifted off a transportation barge or a supply vessel onto the platform; see figure 1.16. It has long been recognized that it is beneficial - from the point of view of the overall construction time and costs of a platform - to be able to reduce the number of modules to be connected up offshore to the smallest possible number. This has resulted in a tremendous escalation in the lifting capacity of floating cranes.

Figure 1.16: Semi-Submersable Crane Vessel

Until the mid-sixties, the heavy lift vessels for the offshore industry were either converted tankers fitted with a revolving crane at the bow or conventional flat barges with a fixed gantry type crane. Their lifting capacity amounted a few hundred tons. After the mid seventies, a new breed of heavy lift vessels was introduced, which consists of Semi-Submersible Crane Vessels (SSCV’s) with a large displacement. They were fitted with two revolving cranes, which could work independently or in tandem depending on the work required. At present, crane vessels are in service which can lift masses of over 10 thousand tons by
using the two cranes simultaneously. Due to the superior motion behavior of the semi-
submersible hull form in waves, the workability with respect to crane operations increased
tremendously.

Subjects of prime importance for crane operations offshore are the impact load that can
occur as a load is taken off a transportation barge, because of the different vertical velocities
of the two vessels. In an environment with waves the barge and the heavy lift crane vessel
carry out wave frequency motions. As the lift is taken off the barge, it will carry out
vertical motions which are dictated by the crane vessel motion characteristics. The barge,
on the other hand, also carries out wave frequency motions which - in general - will not
be in phase with the motions of the load in the hook. When the load is just clearing the
barge, this will result in impacting of the load on the barge. This may lead to damage of
the load, the barge or the crane vessel or even all three of them. The prime concern then
is to reduce the risk that a critical impact occurs. This can be influenced by the design of
the crane vessel, for instance by including the ability to transport the load on the deck of
the crane vessel, but also by operational procedures and aids such as quick de-ballasting
systems. Besides the occurrence of impact loads which are mainly affected by the vertical
motion response to waves, the horizontal motions of the crane vessel are of importance
with respect to the feasibility of accurate positioning of the load onto the platform under
construction. These motions are affected by environmental loads due to wind, waves and
current acting on the crane vessel, by the design and layout of the mooring or positioning
system and by the dynamics of the load hanging in the crane.

Thus, aspects of importance or interest of crane vessels are:
- the vertical crane hook motions in waves,
- the horizontal positioning accuracy of the hook load,
- the impacts on lift-off and set-down of the load,
- the deck load capacity,
- the transit speed and
- the mooring forces.

For the DP system extensive investigations have been carried out into the following aspects
of the station keeping ability:
- the thruster-hull, thruster-thruster and thruster-current interactions,
- the drift forces due to waves,
- the current forces and
- the wind forces.

1.2.6 Transportation Vessels

Some types of transportation vessels are discussed here: shuttle tankers, heavy lift trans-
portation vessels, launch barges, ocean-going tugs, supply vessels and personnel transfer
vessels.

Shuttle Tankers

From the storage facilities the oil is transported to shore by means of pipelines or shuttle
tankers. These shuttle tankers are temporarily moored close to the storage system by
means of a single point mooring (SPM) system, mooring in tandem (see figure 1.17) or
alongside the storage vessel or by means of a DP system.
It may be noted that seamen on a shuttle tanker don’t like mooring in tandem behind an FPSO. Weather vaning of both vessels causes that their ship is in the smoke of the FPSO continuously.

**Heavy Lift Transportation Vessels**

Heavy lift transportation vessels are towed or self-propelled heavy and voluminous cargo barges which are used to transport large volume heavy lifts over long distances. The cargo is taken aboard by either roll-on from ashore or float-on by submerging the cargo deck. Self-propelled heavy lift transportation vessels have been developed; firstly based on the conversion of conventional bulk cargo vessels or tankers and later by purpose built vessels; see figure 1.18. Some of these vessels can even transport dry-docks in which they can dock themselves. The advantage of these vessels is that they can be used to transport drilling semi-submersibles and jack-ups over long distances in a relatively short time, with speeds in a seaway up to about 13 knots. The alternative would be a relatively slow process of towing such structures by means of conventional ocean-going tugs. During the voyage, often use will be made of advises of weather routing offices.

Aspects of importance or interest of heavy lift transportation vessels are:
- the transit speed,
- the motion behavior of the vessel with large heavy cargo on its deck,
- the sea-fastenings to secure the cargo to the vessel and
- the wave impact loads on the cargo, which often overhangs the deck edge of the vessel.

Careful assessment of the motions and accelerations of the vessel in laden condition can lead to considerable savings in the sea-fastenings.

**Launch Barges**

Launch barges are very large flat top barges which transport the jacket structures and other construction elements of a platform which have been built onshore; see figure 1.19. The barge is towed out to the work location.

Aspects of interest during towing of a launch barge are:
- the towing resistance of the barge,
- the course stability under tow,
1.2. PROBLEMS OF INTEREST

- the motions and accelerations of the barge and
- the forces on sea-fastenings holding the jacket on the barge.

At the work location, the barge is held stationary, while the jacket structure is end-launched off the barge. During this launching operation - see figure 1.7 - the barge is not tightly held by the tugs since, as the jacket is launched, the barge will carry out relatively violent motions and be pushed away over quite a distance. The launching operation is initiated after the sea-fastenings have been cut loose and the jacket is only resting on the launchways. By ballasting the barge to the specified trim angle and with initial winch pulling, the jacket will start to slide over the launchways.

Aspects of importance or interest for the barge during launching are:
- the maximum depth of submergence of the aft end of the barge,
- the ballast distribution,
- the friction of the launchways,
- the angle of rotation of the rocker beams,
- the rocker beam forces and
- the stability during the launch.

Very slender structures are sometimes side-launched. Some jackets have been designed to be crane-lifted off the barge by a crane barge or a crane vessel. Then the impact loads on jacket and barge at the moment of lift-off is an aspect of importance.

Tugs and Supply Vessels

Tugs and supply vessels are the 'workhorses' of the offshore industry. Tugs are used to tow all types of floating offshore equipment out to the work location and are in many cases a permanent 'companion'; see figure 8.32. Large deep-sea tugs are used for long distance towing and smaller tugs are employed for short range towing operations. A special type of tug is the so-called anchor handling tug. This vessel is used to position and re-position the anchors of drilling rigs and in stand-by mode it is used for a pipe laying vessel as it proceeds along the pipe line trajectory.
CHAPTER 1. INTRODUCTION

Supply vessels do as their name implies; they are used primarily to transport equipment necessary for the execution of work offshore. This can mean, for instance, drill pipes and drilling muds for a drilling vessel or pipe sections for a pipe laying vessel. The design of supply vessels reflect that they are often dedicated to operations in a particular area. A typical aspect of the design of a supply vessel is the forward location of the deck house with the large open space aft. All equipment to be transported is located on the aft deck; see figure 1.21.

The seakeeping characteristics of these vessels are very important. Aspects of importance or interest of supply vessels and anchor handling tugs are:
- the motions and accelerations on the work deck and on the bridge,
- the vertical relative motions between the waves and the deck edges (deck wetness),
- the effect of shipped water in pipes transported on the open deck of supply vessels,
- the slamming of waves against horizontal hull surfaces and overhanging cargo and
1.2. PROBLEMS OF INTEREST

Figure 1.21: Supply Vessel

- the station keeping near platforms during (un)loading and anchor handling operations.

Personnel Transfer Vessels

Helicopters are the most important means to transfer personnel safely and quickly from and to offshore units and ships. For this, these structures have to be equipped with a helicopter deck. Also, supply vessels are still used to transfer personnel. Figure 1.22 shows a special crew boat for personnel transfer, with a length of 31 meter, which can transport 55 passengers with a cruising speed of 20 knots. However, in the North Sea areas, the helicopter is the most common vehicle for personnel transfer.

Figure 1.22: Crew Boat

For long distances, helicopters are used to transfer pilots to and from ships too. Pilot vessels are used generally for short distances. Conventional types of pilot vessels operate with service speeds under 20 knots. Generally, they are operating safely in weather conditions up to sea states defined by Beaufort 9 or 10. Figure 1.23 shows some modern types of these vessels: a planing hull ship and small waterplane area twin hull (SWATH) ship, with maximum speeds of about 28 knots. At a reduced speed, they can operate safely in weather conditions up to Beaufort 8.
Aspects of importance or interest during sailing of these vessels are:
- the vertical accelerations of the vessel,
- the wave-induced vibrations in the vessel and
- the slamming behavior of the vessel at high speeds.
The wave-induced vibrations and the slamming forces at high speeds can cause considerable (fatigue) damage to these vessels and be dangerous to people.

Aspects of importance or interest during (dis)embarkation a vessel are:
- the maneuverability of the vessel,
- the vertical and horizontal motions and
- the safety of crew and pilots during (dis)embarkation.

Figure 1.23: Pilot Vessels
Chapter 2

HYDROSTATICS

2.1 Introduction

This chapter discusses only the static properties of a structure. It is assumed here that any disturbance to the equilibrium state will be brought about so slowly that all dynamic effects can be ignored.

First, the hydrostatic pressure, Archimedes’ law, the resulting internal forces in a structure and the concept of effective tension will be considered. The effective tension is a fictitious internal force or stress, acting within submerged bodies. It is referred to as effective tension here in order to distinguish it from effective stress as used in soil mechanics - a somewhat different entity.

Static floating stability will be considered later in this chapter. One often takes for granted that a floating structure will float in a proper (desired or upright) position. This will happen only if it is correctly designed, however. A ”good” example is the Swedish sailing vessel VASA, see [Franzen, 1962], which capsized in 1628 on her maiden voyage which lasted less than two hours; she sank in calm water while still in view of the people of Stockholm.

In service a ship or other floating structure will experience many external loads (e.g. from wind and waves) as well as ”internal” loads (from cargo, for example) trying to turn it over. It must be able to resist these, via what is termed its static stability. Too little stability is obviously not desirable; this was the case with the VASA. Too much stability can be undesirable as well because stability affects a ship’s natural roll frequency; excessive and unpleasant motions in a seaway can be the result. Too much stability can be costly too. Thus, as with so many other design features, stability is often a compromise. Because a floating structure will meet varied conditions during its life, its ability to survive should ideally be expressed in statistical terms with stability standards set accordingly. Indeed, no floating structure can be guaranteed to remain stable under all conditions. Even though it is possible to design a boat - such as a motor lifeboat used by a coastal rescue team - to be self-righting; this does not guarantee that it always remains upright!
CHAPTER 2. HYDROSTATICS

2.2 Static Loads

2.2.1 Hydrostatic Pressure

In any body of fluid, hydrostatic pressure results from the weight of the fluid column above the point at which that pressure is measured. At a free water surface, the pressure will normally be zero relative to the atmospheric pressure (which is nearly always neglected in offshore hydromechanics). In equation form:

\[ p = \frac{1}{2} \rho gh \] (2.1)

in which:

- \( g \) = acceleration of gravity
- \( h \) = distance below the fluid surface
- \( p \) = pressure (= force/area)
- \( \rho \) = mass density of the fluid

For sea water, the mass density, \( \rho \), is in the order of 1025 kg/m\(^3\); one may remember from oceanography that this density is a function of temperature and salinity, by the way.

2.2.2 Archimedes Law and Buoyancy

Archimedes - who lived from 285 until 212 BC - is reported to have run naked through the streets after making his discovery that a body submerged in a fluid experiences an upward buoyant force, \( F_\nabla \), equal to:

\[ F_\nabla = \rho g \nabla \] (2.2)

in which:

- \( F_\nabla \) = buoyant force
- \( \nabla \) = volume of the submerged part of the object

This approach views buoyancy as a distributed mass force, completely analogous to the gravity force on (or weight of) the mass of an object.

One can look at buoyancy in another way, however. Since the hydrostatic pressure at any chosen point on the underside of a submerged body will be higher than the pressure on the upper surface directly above that point, one can expect there to be a net upward force on that body. Indeed, an equivalent way of determining \( \rho g \nabla \) is to take the vertical component of the pressure times area and to integrate this function over the entire surface of the body. This buoyant force is also equal (or equivalent) to the integral of the hydrostatic pressure vector over the surface of the body.

2.2.3 Internal Static Loads

The two different ways of looking at the buoyant force on a body lead to two different approaches to static internal loads on a structure as well. This is best illustrated by considering a steel drilling pipe hanging in an oil well as shown schematically in figure 2.1.
2.2. STATIC LOADS

Figure 2.1: Forces and Pressures on A Vertical Drill String

The string is hanging with its lower end above the bottom of the vertical well which is further filled only with air (for this intellectual exercise). Because its own weight - the gravity force - is a distributed force (along the length in this case), the tension force in the drill string will depend upon location along its length as well. The tension will be maximum at the top of well \((z = 0)\) and will reduce linearly to zero at the bottom end of the string \((z = L)\). In equation form, this tension is given by:

\[
T = \frac{1}{2} \rho_s g A (L - z) \tag{2.3}
\]

in which:

\[
\begin{align*}
L &= \text{Length of the pipe hanging in the well (m)} \\
g &= \text{Acceleration of gravity (m/s}^2) \\
A &= \text{Cross section area of steel in the pipe (m}^2) \\
z &= \text{Coordinate measured downward from the top in this special case! (m)} \\
\rho_s &= \text{Mass density of steel (kg/m}^3\)
\end{align*}
\]

This tension is shown graphically as line \(AB\) in the figure and should be quite straightforward; confusion can result after the well is filled with a liquid such as drilling mud.

If the buoyant force caused by the mud is treated as a distributed force just as did Archimedes, then one could conclude that there is a distributed buoyant force acting on the entire body equal to the weight of drilling mud displaced by the drill string. Starting at the bottom (considering only a thin (horizontal) slice of the string first), the buoyant force on this slice will be (nearly) zero simply because its volume is so small. As one considers an ever-increasing string slice length, this buoyant force will increase to:

\[
\rho_m g A L \tag{2.4}
\]
at the top of the string. Since this buoyant force acts in a direction opposite to the gravity force, one finds:

\[ T_e = (\rho_s - \rho_m)gA(L - z) \] 

(2.5)

Since \( \rho_s > \rho_m \), \( T_e \) will be generally be greater than zero as well, but will be less than \( T \), above. \( T_e \) is often referred to as the effective tension. It, too, varies linearly from a maximum value at \( z = 0 \) to zero at \( z = L \) and indicated by line \( EB \) in the figure. In general the effective tension is based upon the submerged weight of the body. This is the weight of the body in air minus the buoyant force acting on it. It can be computed for homogeneous bodies in water using a density of \( (\rho_b - \rho) \), where \( \rho_b \) is the mass density of the body.

If, on the other hand, one treats the buoyant force as the resultant of pressures acting on the surface of the body - the drill string in this case, then one finds first that the pressures on the (vertical) drill string walls are in equilibrium and contribute no vertical force. The drill string does experience a vertical force at its bottom end which is equal to:

\[ F_z = \rho_m gAL \] 

(2.6)

This concentrated vertical upward force combines with the distributed weight of the drill string by shifting line \( AB \) in the figure to position \( DE \). One discovers now that the lower section of the drill string is in compression instead of tension! The length that is in compression can be computed:

\[ L_c = \frac{\rho_m}{\rho_s}L \] 

(2.7)

One can also conclude that there will be no axial force in the drill string at an elevation \( L_c \) above its bottom end and that the entire length below that point will be in compression. This does not agree with what was found when the buoyant force was treated as Archimedes had done, above.

It is easiest to resolve this apparent dilemma by examining a slightly different situation. Consider a solid body which has the same density as water; an athletic swimmer is a reasonable approximation of this. Considering buoyancy as a distributed force (in the Archimedian sense) would now yield an effective tension in this swimmer’s body that would be zero, independent of the depth to which he or she might happen to dive. As anyone who has suddenly plunged into a pool has probably discovered, this is not the case. Indeed, as one dives rapidly deeper into the water the external pressure trying to crush one’s body only gets larger. This observation supports the external pressure form of buoyancy argument rather than Archimedes’ approach. Apparently, the external pressure form is better for revealing the actual true stress in the submerged body, just as with the rapidly diving swimmer.

On the other hand, the effective tension - resulting from the distributed force approach to buoyancy - also has its uses even though the effective tension will not be measured by a strain gauge mounted on the submerged object. Consider a given length of metal chain hanging between two fixed points in air. The tension at any point in that chain is dependent upon the chain’s weight per unit length. If the chain were hung in the same configuration under water, one would expect the tension to be lower. That will indeed be the case: the tension force between adjacent links of the chain now becomes the effective tension.
2.2.4 Drill String Buckling

Returning to the hanging drill string problem posed above, an important structural engineering question involves the buckling behavior of its lower part. It was shown above that the actual axial force in the lower $L_c$ meters of the drill string is compressive. Since the mud surrounding this string provides no lateral support, a structural engineer would feel very justified in applying the Euler column buckling formula:

$$\frac{P_{cr}}{A} = \frac{\pi^2 E}{(L_c/r)^2}$$

(2.8)

in which:

- $P_{cr}$ = Critical buckling load (N)
- $A$ = Cross section area of the column (m$^2$)
- $E$ = Material elastic modulus (N/m$^2$)
- $L_c/r$ = Column slenderness ratio (-)

to determine that if $L_c$ is long enough, the drill string segment should buckle.

Another structural engineer can argue that the effective tension in this drill string segment is positive over its entire length; a structure in tension will not buckle.

An independent approach is needed to resolve this discussion. An independent theoretical approach follows from potential energy changes. How does the potential energy of the mud plus drill string change as the drill string buckles? Considering first the drill string steel, if it buckles, then the bottom end of the drill string will move up very slightly, thus raising its center of gravity and thereby increasing the total potential energy of the string. The mud, on the other hand, loses a minute amount of potential energy as it compensates for this drill string movement. Since the drill string has a higher density than the mud, the string gains more potential energy than the mud loses. Potential energy is thus created if the drill string buckles! Any type of engineer should recognize that this cannot happen; One cannot expect the drill string to buckle when it is hanging freely in the mud-filled well.

Apparently two conditions must be met before a long slender submerged object will buckle:
- it must actually be under compression and
- its effective tension must be negative.

These criteria can be checked by considering a chain instead of the drill string hanging in the well. The positive weight minus buoyant force of each link will cause there to be a positive effective tension between each link. However, since each link is also essentially surrounded by drilling fluid exerting an external pressure on its surface, it should be equally obvious that the material in each of the deepest links (at least) will be experiencing a compressive force, just like the diving swimmer, above.

Another check can be made by considering - at least intellectually - a drilling string with a density which is less than that of the drilling mud. One can see immediately from equation 2.5 that the effective tension will now be negative. Indeed, the sting will buckle as soon as a long enough segment is lowered into the drilling mud.

Of course the force situation in the drill string changes as soon as a point of a drill bit tooth is lowered to the hole bottom. From that moment on, an extra upward force is exerted on the base of the drill string so that the effective tension becomes negative in the lowest part
of the string. This negative effective tension segment gets longer in direct proportion to
the well bottom contact force. If this contact force gets big enough, then the lower segment
of the drill string will obviously buckle.

An experimental test of drill string buckling has been carried out by D. Brussmann in
Germany. Unfortunately, his work is not reported in the open literature, but one of the
authors has seen photos of his test. The photos show - in contrast to the conclusions
above - that the drill string will buckle even before it touches the bottom of the well. This
obviously makes the confusion complete and can initiate an interesting discussion of what
to believe of all this.

More than a century ago, a famous hydraulic engineer (in his time), C.M. Allen is reported
to have remarked: "When the theory and the experiment do not agree, then take a good
hard look at the theory." In any case, a safe conclusion to all this is that new - confirmatory
tests are needed. These are being carried out at the Delft University of Technology.

2.2.5 Pipeline on Sea Bed

Consider the horizontal force balance on an isolated segment of a horizontal pipe submerged
in (and completely surrounded by) a fluid; the fluid pressure acting at the depth of the
center line of the pipe is $p$. The mathematics is simplest if one makes two simplifications:

1. The pipe has a rectangular cross section with height 1; this makes all forces either
   horizontal or vertical.

2. The pressure change between the top and bottom of the pipe - at least that on the
   vertical surfaces of the pipe - can be neglected. This is not acceptable if the vertical
   equilibrium is considered; it is the pressure difference the top and bottom surfaces
   which yields the buoyant force. These vertical forces have no effect on the horizontal
   equilibrium of interest here, however.

Consider first a straight pipe. The pressure forces on each side of the pipe will be opposite,
equal and have identical lines of action; they will cancel. The same will be true of the
hydrostatic pressure forces on the pipe ends. Of course, all these external pressures will
cause compressive loadings within the pipe wall, but there is no resultant horizontal force
and no (internal) moment in a straight segment of an isolated horizontal pipe (or bar).
This means that no matter how long the structure is, and even though it is obviously
subjected to a potentially large compressive load, no moments are generated. This is quite
like the case with the vertical drill string. The effective tension in this pipe is now zero
and it does not buckle. This is in agreement with the two 'rules' given above.

Even if the pipe has an initial curvature, one can work out the pressure force balance to
discover that there is still no resulting moment in the pipe; it still will not buckle! Instead,
just as one might expect when the limit $T_e = 0$ is reached in the above 'rules', the pipe has
come to a neutral state of equilibrium.

2.3 Static Floating Stability

The static stability of a floating structure encompasses the up-righting properties of the
structure when it is brought out of equilibrium or balance by a disturbance in the form
of a force and/or moment. As a result of these (extra) loads, the structure will translate and/or rotate about its center of gravity. Formally, dynamic as well as static properties of the structure play a role in this. In this chapter - as mentioned above - only the static properties of the structure will be considered. Dynamic effects are considered in later chapters of this book, however.

### 2.3.1 Definitions

The body axes and the notations, as used here in static stability calculations, are presented in figure 2.2.

![Figure 2.2: Body Axes and Notations](image)

Planes have been defined here as they are generally used in ship hydrostatics. Most ships have only one plane of symmetry, called the middle line plane \((x, z)\); this vertical plane is the principal plane of reference. The design water plane or load water plane (simply called water plane here) is a plane perpendicular to the middle line plane, chosen at the still water surface. The base plane of the structure is a plane perpendicular to the middle line plane through the keel or the bottom of the ship. It may or may not be parallel to the water plane in the longitudinal direction. Planes perpendicular to both the middle line plane and the water plane are called transverse planes. A transverse plane at half the length of the ship is called the amidships section of the ship.

So-called hydrostatic forces and moments, caused by the surrounding water, will act on a structure in still water. It is known from above that the buoyancy of a structure immersed in a fluid is the vertical upthrust that the structure experiences due to the displacement of the fluid.

![Figure 2.3: Definition of Centers and Forces](image)

The center of the volume of the fluid displaced by a floating structure is known as the center of buoyancy \(B\), see figure 2.3-a. The first moment of the under water volume about the center of buoyancy is zero.
The projections of the center of buoyancy $B$ of a ship in the plan and in the section are known as the longitudinal center of buoyancy ($LCB$) and the vertical center of buoyancy ($VCB$). If the structure is not symmetrical below the water plane, then the center of buoyancy will not be situated in the middle line plane. Its projections in plan may then be referred to as the transverse center of buoyancy ($TCB$).

Defining it formally: The center of gravity, $G$, of a structure is that point through which, for static considerations, the whole weight of the structure may be assumed to act, see figure 2.3-b. The first moment of mass or weight about the center of gravity is zero. The center of volume of its immersed part also defines the center of buoyancy, $B$.

For the sake of simplicity and understanding, the disturbances and the hydrostatic forces and moments are considered (in this text) only to act in the plane of the drawing. The structure is considered to be a floating cylinder with a constant but arbitrarily shaped cross section. The center of gravity $G$ and the center of buoyancy $B$ of the structure are assumed to be positioned in the plane of drawing.

Rotations in the plane of drawing are defined here as heel, a rotation about the structure’s longitudinal horizontal axis. The same principles holds as well for trim, a rotation about the body’s transverse horizontal axis. Superposition can be used for combinations of heel and trim - at least if the angles of rotation are not too large.

### 2.3.2 Equilibrium

In figure 2.3, $m$ is the mass of the floating structure. Often in literature on ship and offshore hydromechanics, this displacement mass:

$$m = \rho V$$

(Archimedes’ Law) (2.9)

is denoted sometimes by the symbol $\Delta$ (delta).

A floating structure is said to be in a state of equilibrium or balance when the resultant of all the forces acting on it is zero and the resulting moment of these forces is also zero. Three different states of equilibrium or types of stability can be distinguished for a structure which is subject to a small disturbance from an equilibrium position (see figure 2.4):

1. If, following the disturbance, the structure tends to return to the equilibrium position it is said to be in a state of stable equilibrium or to possess positive stability.

2. If, following the disturbance, the structure remains in its new position, then it is said to be in a state of neutral equilibrium or to possess neutral stability.

3. If, following the disturbance, the excursion from the equilibrium position tends to increase, the structure is said to be in a state of unstable equilibrium or to possess negative stability.

If a structure is floating freely in rest in a fluid, the following equilibrium or balance conditions in the plane of the drawing in figure 2.3 are fulfilled:

1. **Horizontal equilibrium:** the sum of the horizontal forces equals zero.

2. **Vertical equilibrium:** the sum of the vertical forces equals zero.

3. **Rotational equilibrium:** the sum of the moments about $G$ - or any other point - equals zero.
2.3. STATIC FLOATING STABILITY

Horizontal Equilibrium
A translation in a horizontal direction leads to no resultant hydrostatic force, so that unless other external influences - such as moorings are involved - the structure is in neutral equilibrium for this type of disturbance. This equilibrium is of no further interest here.

Vertical Equilibrium
For a floating structure, a vertical downward movement (sinking deeper) results in an increase of the buoyant force which will tend to force the structure back upwards; it tends to return the structure to its original state of equilibrium so that the structure is stable for this type of disturbance.
Archimedes’ principle holds for the vertical equilibrium between buoyancy and gravity forces:
\[ \rho g \nabla = gm \] (2.10)
If an additional mass, \( p \), is placed on this structure, its original equilibrium will be disturbed. The structure will sink deeper and heel until a new state of equilibrium has been reached. The new vertical equilibrium is given by:
\[ \rho g \cdot (\nabla + \Delta \nabla) = g \cdot (m + p) \] (2.11)
in which \( \Delta \nabla \) is the increase of the volume of displacement of the floating structure.
If the mass \( p \) has been placed on the structure in such a manner that it only sinks deeper parallel to the water plane without heel, the change of draft \( \Delta T \) follows from:
\[ \Delta \nabla = \Delta T \cdot A_{WL} = \frac{p}{\rho} \quad \text{or:} \quad \Delta T = \frac{p}{\rho \cdot A_{WL}} \] (2.12)
Here, \( A_{WL} \) is the area of the water plane and it is assumed that this area is constant over the draft interval \( \Delta T \).

Rotational Equilibrium
A moment acting on the structure results in a rotation about \( G \), or heel. The structure may display a stable, a neutral or even an unstable equilibrium.
If an external heeling moment acts on the structure as given in figure 2.5, it follows from the rotational equilibrium:
\[ M_H = \rho g \nabla \cdot y = gm \cdot y \] (2.13)
From this follows too that if no external moment acts on the structure, the lever arm $y$ should be zero:

$$M_H = 0 \text{ which results in: } y = 0$$

(2.14)

This means that for any floating structure at rest, the center of buoyancy $B$ and the center of gravity $G$ will be situated on the same vertical line. If this is not so, the structure will heel or trim until they do become vertically aligned. This means too that the longitudinal position of the center of gravity can be found easily from the longitudinal position of the center of buoyancy, which can be derived from the under water geometry of the structure.

2.3.3 Shifting Masses and Volumes

Static stability problems are mainly related to an addition, a removal or a shift of masses and/or volumes (of displacement). The principal effects of these actions are a shift of the total center of gravity, $G$, and/or of the center of buoyancy, $B$. These effects will be treated here first.

Consider a structure with a mass $m$. This mass includes a mass $p$, placed somewhere on the structure. When this mass, $p$, will be shifted now over a certain distance, $c$, as shown in figure 2.6, the original overall center of gravity $G_0$ will be shifted to $G_1$ - parallel to this displacement - over a distance equal to:

$$\overline{G_0 G_1} = \frac{p \cdot c}{m}$$

(2.15)

This can be seen easily by writing the first moments of the masses with respect to the $x$- and $y$-axes, respectively:

$$m \cdot x_{G_1} = m \cdot x_{G_0} + p \cdot (x_{p_1} - x_{p_0})$$

$$m \cdot y_{G_1} = m \cdot y_{G_0} + p \cdot (y_{p_1} - y_{p_0})$$

new = old + change

(2.16)

Then, the shift of the center of gravity of the structure from $G_0$ to $G_1$ is given by:
2.3. **STATIC FLOATING STABILITY**

Combining these two equations provides equation 2.15 above.

From these two equations follows too that:

\[
x_{G_1} - x_{G_0} = \frac{p \cdot (x_{p_1} - x_{p_0})}{m}
\]

\[
y_{G_1} - y_{G_0} = \frac{p \cdot (y_{p_1} - y_{p_0})}{m}
\]

(2.17)

which means that the shift of \( G_0 \) to \( G_1 \) will be parallel to the shift of the mass \( p \).

In case of a freely suspended mass, \( p \), (cargo hanging in a crane of a ship is a good example) this mass \( p \) should be considered as being concentrated at the suspension point of the crane hoisting rope. This becomes the case immediately after hoisting the cargo (assuming that the suspended cargo is not horizontally restrained in any way). The vertical elevation of the cargo above or beneath the deck or a horizontal displacement of the mass by a changed angle of heel of the ship is of no importance at all on the ship’s center of gravity. At each angle of heel, the cargo acts as a vertical force acting downwards at the suspension point in the crane.

One can also discover that the center of buoyancy shifts parallel to a line through the centers of the volumes of the emerged and the immersed water displacement wedges when a floating body heels by an external moment only. The volume of the emerged wedge, in fact, has been shifted to the immersed wedge; see figure 2.7.

Because these two volumes are equal, two succeeding water planes with a small mutual difference in angle of heel intersect each other on a line with respect to which the first moments of volume of the two wedges are zero. This is a line through the center of the water plane. This means that the structure heels and/or trims about a line through the center of the water plane, the *center of floatation*. In case of a heeling ship (with symmetric water planes) this is a line at half the breadth of the water plane.
2.3.4 Righting Moments

A structure is floating at rest. Because of the rotational equilibrium, the center of gravity, $G$, is positioned on a vertical line through the center of buoyancy, $B$. If one adds an (external) heeling moment $M_H$ to this structure, it will heel with an angle $\phi$; see figure 2.8.

As a result of this heeling, the shape of the under water part of the structure will change; the center of buoyancy shifts from $B$ to $B_\phi$ on a line parallel to the line through the centers of the emerged and immersed wedges $z_e$ and $z_i$. An equilibrium will be achieved when the righting stability moment $M_S$ equals the (external) heeling moment $M_H$:

$$M_S = M_H$$

in which the righting stability moment, $M_S$, is defined by:

$$M_S = \rho g V \cdot \overrightarrow{GZ}$$
2.3. STATIC FLOATING STABILITY

The righting stability lever arm, $GZ$, can be written as:

$$GZ = GN_\phi \cdot \sin \phi$$  \hspace{1cm} (2.21)

With this, the righting stability moment, $M_S$, becomes:

$$M_S = \rho g \nabla \cdot GN_\phi \cdot \sin \phi$$  \hspace{1cm} (2.22)

The so-called **metacenter**, $N_\phi$, is the point of intersection of the lines through the vertical buoyant forces at a zero angle of heel and at an angle of heel, $\phi$. The position of this metacenter $N_\phi$ depends on the new position of the center of buoyancy, $B_\phi$; so on the shape of the structure at and near its water plane. This zone is sometimes referred to as the "zone between water and wind". The angle of heel $\phi$ and the shape of the emerged and immersed wedges control the shift of the center of buoyancy from $B$ to $B_\phi$ and thereby also the position of $N_\phi$.

### 2.3.5 Metacenter

This metacenter, $N_\phi$, will be explained here in detail for the case of a rectangular barge. When using this barge - with length $L$, breadth $B$ and draft $T$ - the volume of displacement, $\nabla$, and the center of buoyancy, $B$, can be determined easily. Then, the emerged and immersed wedges in the cross sections are bounded by vertical lines so that these wedges are right angle triangles and the position of its centroids can be calculated easily too. The barge is floating in an upright even keel condition. An external heeling moment, $M_H$, causes the pontoon to heel to a relatively large angle, $\phi$. The position of the metacenter $N_\phi$ can be calculated easily in such a special case; see figure 2.9.

![Figure 2.9: Rectangular Barge Stability](image)

The shift of $z_e$ of the emerged wedge to $z_i$ of the immersed wedge can be split in two parts:
1. A horizontal shift \( z_e z_i \).

This horizontal shift causes a horizontal displacement of the center of buoyancy from \( B \) to \( B' \):

\[
BB' = BM \cdot \tan \phi
\]  

(2.23)

and the vertical buoyancy force, \( \rho g \), intersects the vertical buoyancy force at \( \phi = 0^\circ \) at the initial metacenter \( M \).

The first moment of volumes with respect to the middle line plane of the barge in the heeled condition is given by:

\[
\{LBT\} \cdot \{BM \tan \phi\} = \{LBT\} \cdot \{0\} + 2 \cdot \left\{ \frac{L \cdot B}{2} \cdot \frac{B}{2} \cdot \tan \phi \right\} \cdot \left\{ \frac{2B}{3} \right\} \tag{2.24}
\]

so that:

\[
BM = \frac{1}{12} \cdot L \cdot B^3 \quad \frac{L \cdot B}{2} \cdot T \tag{2.25}
\]

or expressed in terms of the moment of inertia (second moment of areas) of the water plane, \( I_T \), and the displacement volume of the barge \( \nabla \):

\[
BM = \frac{I_T}{\nabla} \tag{2.26}
\]

2. A vertical shift \( z_i z_i \).

This vertical shift causes a vertical displacement of the center of buoyancy from \( B' \) to \( B \):

\[
B'B = MN \tag{2.27}
\]

and the vertical buoyancy force, \( \rho g \), intersects the vertical buoyancy force at \( \phi = 0^\circ \) at the metacenter \( N \).

The first moment of volumes with respect to a plane parallel to the not heeled water plane through the center of buoyancy \( B \) of the barge in the not heeled condition is given by:

\[
\{LBT\} \cdot \{MN \} = \{LBT\} \cdot \{0\} + 2 \cdot \left\{ \frac{L \cdot B}{2} \cdot \frac{B}{2} \cdot \tan \phi \right\} \cdot \left\{ \frac{1}{3} \cdot \frac{B}{2} \cdot \tan \phi \right\} \tag{2.28}
\]

so that:

\[
MN = \frac{1}{12} \cdot L \cdot B^3 \cdot \frac{1}{2} \tan^2 \phi \tag{2.29}
\]

or again expressed in terms of the moment of inertia of the water plane, \( I_T \), the displacement volume of the barge \( \nabla \) and the angle of heel \( \phi \):

\[
MN = \frac{I_T}{\nabla} \cdot \frac{1}{2} \tan^2 \phi \tag{2.30}
\]
Now the position of the metacenter $N_\phi$ follows from a superposition of equations 2.26 and 2.30:

$$\overline{BN_\phi} = \overline{BM} + MN_\phi$$
$$= \frac{I_T}{\nabla} \left(1 + \frac{1}{2}\tan^2\phi\right) \quad (2.31)$$

or:

$$\overline{BN_\phi} = \overline{BM} \cdot \left(1 + \frac{1}{2}\tan^2\phi\right) \quad (2.32)$$

The term $\overline{BM} \cdot 1$ represents the effect of the horizontal part of the shift of $z_c$ to $z_i$ and the term $\overline{BM} \cdot \frac{1}{2}\tan^2\phi$ represents the effect of the vertical part of this shift. Note that the angle of heel, $\phi$, has no effect on $M$; but it has effect on $N_\phi$.

### 2.3.6 Scribanti Formula

![Figure 2.10: Wall Sided Structure](image)

A floating structure is said to be **wall-sided** if, for the angles of heel to be considered, those portions of the hull covered or uncovered by the changing water plane are vertical when the structure is floating upright; see figure 2.10. The wedges in the cross sections are bounded by vertical lines - as is the case for the barge - so that the emerged and immersed wedges are right angle symmetrical triangles and the position of the metacenter $N_\phi$ can be calculated easily in such a special case.

It can be found that - regardless the under-water geometry of a structure - equation 2.30 is valid for all wall-sided structures:

$$\overline{BN_\phi} = \frac{I_T}{\nabla} \cdot \left(1 + \frac{1}{2}\tan^2\phi\right) \quad \text{Scribanti Formula} \quad (2.33)$$
in which $I_T$ is the transverse moment of inertia (second moment of areas) of the not heeled water plane about the axis of inclination for both half water planes. This is a fairly good approximation for ships that have vertical sides over a great part of the length and which are not so far heeled that the deck enters the water or the bottom comes above the water level.

At small angles of heel (up to about 10 degrees), the effect of the vertical shift of the center of buoyancy, $B_0 \Delta B = MN_\phi$, can be ignored. This can be seen in the Scribanti formula too, because in that case $\frac{1}{2}\tan^2 \phi$ is very small compared to 1.0. The work line of the vertical buoyancy force, $\rho g \nabla$, intersects the work line of the vertical buoyancy force for $\phi = 0^\circ$ at the initial metacenter, $M$, which location is defined by equation 2.26, useful for $\phi < \pm 10^\circ$. Up to angles of heel of about 10 degrees, this is a fairly good approximation of the metacenter of the ships that are almost wall-sided at the "zone between water and wind" over a relatively large part of the length. Sailing yachts for instance, do not fulfil this requirement.

At larger angles of heel, the effect of the vertical shift of the center of buoyancy must be taken into account. The work line of the vertical buoyancy force, $\rho g \nabla$, intersects the work line of the vertical buoyancy force for $\phi = 0^\circ$ at the metacenter, $N_\phi$, which location is defined by equation 2.32, useful for $\phi > \pm 10^\circ$. Use of this formula yields sufficiently accurate results for "non-Scribanti" ships until heel angles of heel of about 20 to 25 degrees. Keep in mind that this formula is not longer valid when the angle of heel gets so large that the water plane changes rapidly. This is the case when the bilge or chine (the "corner" where the sides of a ship meet its bottom) comes out of the water or the deck enters the water, for example. This possibility always has to be checked, when carrying out stability calculations!

It has been shown above that the metacenter is defined by the intersection of lines through the vertical buoyant forces at a heel angle $\phi$ and a heel angle $\phi + \Delta \phi$. Depending on the magnitude of $\phi$ and the increment $\Delta \phi$, two different metacenter definitions have been distinguished here, as tabled below.

<table>
<thead>
<tr>
<th>metacenter point</th>
<th>symbol</th>
<th>$\phi$</th>
<th>$\Delta \phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>initial metacenter</td>
<td>$M$</td>
<td>0</td>
<td>very small</td>
</tr>
<tr>
<td>metacenter</td>
<td>$N_\phi$</td>
<td>0</td>
<td>larger</td>
</tr>
</tbody>
</table>

Note that - for symmetrical under water forms like ships - both the metcenters $M$ and $N_\phi$ are situated in the middle line plane ($\phi = 0$).

The stability lever arm $GZ = G N_\phi \cdot \sin \phi$ is determined by the hydrostatic properties of the submerged structure (form and heel dependent) and the position of the center of gravity of this structure (mass distribution dependent). This is the reason why the following expression for $G N_\phi$ has been introduced:

$$GN_\phi = KB + BN_\phi - KG$$  \hspace{1cm} (2.34)

Here, $K$ is the keel point of the structure; see figure 2.10. The magnitude of $KB$ follows from the under water geometry of the structure and the magnitude of $KG$ follows from the mass distribution of the structure.
2.3. STATICS FLOATING STABILITY

If, within the range of considered angles of heel, the shape of the water plane does not change very much (wall-sided structure), a substitution of equation 2.32 in equation 2.34 results in:

\[
GN_{\phi} = KB + BM \cdot \left(1 + \frac{1}{2} \tan^2 \phi \right) - KG
\]

\[
= GM + \frac{1}{2} BM \cdot \tan^2 \phi
\]  

(2.35)

For small angles of heel, the stability lever arm becomes

\[
GZ = GM \cdot \sin \phi
\]

and

\[
GM = K B + BM - KG
\]

(2.36)

**Submerged Structures**

![Figure 2.11: Submerged Floating Structure](image)

Fully submerged structures, such as tunnel segments during installation or submarines (see figure 2.11), have no water plane. The definitions of \( BM \) and \( BN_{\phi} \) show that these values are zero, which means that the metacenter coincides with the center of buoyancy. In this case the previous equations reduce to:

\[
GN_{\phi} = GM = KB - KG
\]

(for fully submerged bodies only)  

(2.37)

### 2.3.7 Stability Curve

If a floating structure is brought under a certain angle of heel \( \phi \), see figure 2.12, then the righting stability moment is given by:

\[
M_S = \rho g V \cdot \overline{GZ}
\]

\[
= \rho g V \cdot GN_{\phi} \cdot \sin \phi
\]

\[
= \rho g V \cdot \{GM + MN_{\phi}\} \cdot \sin \phi
\]  

(2.38)

In these relations:

\[
\overline{GZ} = GN_{\phi} \cdot \sin \phi = \{GM + MN_{\phi}\} \cdot \sin \phi
\]

(stability lever arm)  

(2.39)
CHAPTER 2. HYDROSTATICS

The value $GZ$ determines the magnitude of the stability moment. For practical applications it is very convenient to present the stability in the form of righting moments or lever arms about the center of gravity $G$, while the floating structure is heeled at a certain displacement, $\phi$. This is then expressed as a function of $\phi$. Such a function will generally look something like figure 2.13 and is known as the static stability curve or the $GZ$-curve.

Because the stability lever arm is strongly dependent on the angle of heel, $\phi$, a graph of $GZ$, as given in figure 2.13 is very suitable for judging the stability. For an arbitrarily (non symmetric) floating structure form, this curve will not be symmetrical with respect to $\phi = 0$, by the way.

For symmetric forms like ships however, the curve of static stability will be symmetric with respect to $\phi = 0$. In that case, only the right half of this curve will be presented as in figure 2.14.

The heel angle at point $A$ in this figure, at which the second derivative of the curve changes sign, is roughly the angle at which the increase of stability due to side wall effects (Scribanti
2.3. STATIC FLOATING STABILITY

formula) starts to be counteracted by the fact that the deck enters the water or the bilge comes above the water.

Figure 2.15 shows the static stability curve when the initial metacentric height, $GM$, is negative while $GZ$ becomes positive at some reasonable angle of heel $\phi_1$, the so-called angle of loll.

If the floating structure is momentarily at some angle of heel less than $\phi_1$, the moment acting on the structure due to $GZ$ tends to increase the heel. If the angle is greater than $\phi_1$, the moment tends to reduce the heel. Thus the angle $\phi_1$ is a position of stable equilibrium. Unfortunately, since the $GZ$ curve is symmetrical about the origin, as $\phi_1$ is decreased, the floating structure eventually passes through the upright condition and will then suddenly lurch over towards the angle $-\phi_1$ on the opposite side and overshoot this value (because of dynamic effects) before reaching a steady state. This causes an unpleasant rolling motion, which is often the only direct indication that the heel to one side is due to a negative $GM$ rather than a positive heeling moment acting on the structure.
Characteristics of Stability Curve

Some important characteristics of the static stability curve can be summarized here:

1. **Slope at The Origin**
   For small angles of heel, the righting lever arm is proportional to the curve slope and the metacenter is effectively a fixed point. It follows, that the tangent to the $GZ$ curve at the origin represents the metacentric height $GM$. This can be shown easily for the case of a wall-sided structure:

   \[
   \frac{d}{d\phi} \{GZ\} = \frac{d}{d\phi} \{GN_\phi \cdot \sin \phi\} = \frac{d}{d\phi} \left\{ \left( GM + \frac{1}{2}BM \cdot \tan^2 \phi \right) \cdot \sin \phi \right\} \tag{2.40}
   \]

   This derivative becomes $GM$ for zero heel. This means that the initial metacentric height $GM$ can be of great importance for the further form of the curve, especially at smaller angles of heel, and for the area under the curve (see item 5 below).

2. **Maximum $GZ$ Value**
   The maximum $GZ$ value is rules the largest steady heeling moment that the floating structure can resist without capsizing. Its value and the angle at which it occurs are both important. The shape of the above-water part of the floating structure is of great importance for the maximum attainable value of the stability lever arm.

3. **Range of Stability**
   At some angle of heel (sometimes even greater than 90 degrees) the $GZ$ value decreases again to zero and even becomes negative for larger inclinations. This angle is known as the angle of vanishing stability. The range of angles for which $GZ$ is positive is known as the range of stability. This range is of great importance for the maximum attainable area under the stability curve and thereby also on the maximum potential energy that the structure can absorb via a roll motion. The shape of the above-water part has a large influence on the angle of vanishing stability; compare curves I and II in figures 2.12 and 2.13. For angles within the range of stability, a floating structure will return to the upright state when the heeling moment is removed.

4. **Angle of Deck Edge Immersion**
   For most floating structures, there is a point of inflection in the stability curve, corresponding roughly to the angle at which the deck edge becomes immersed. This point is not so much of interest in its own right as in the fact that it provides guidance to the designer upon the possible effect of certain design changes on stability. The shape of the above water part of the structure can have a large influence on its static stability. More or less the same statement can be made when the bilge or the bottom chine emerges, because of the decrease of the breadth of the water line. Keep in mind that for wall-sided structures, when the deck enters the water or the bottom chine comes above the water level, the immersed and emerged wedges are no longer nice triangles; calculations become much more cumbersome!
5. **Area Under The Static Stability Curve**

   An important parameter when judging the stability properties of a structure floating upright is the work that has to be done to reach a chosen heel angle, $\phi^*$:

   \[
   P_{\phi^*} = \int_{0}^{\phi^*} M_S \cdot d\phi
   \]

   \[
   = \rho g \nabla \cdot \int_{0}^{\phi^*} G N_0 \cdot \sin\phi \cdot d\phi
   \]  
   \[
   \text{(2.41)}
   \]

   This means that the area under the static stability curve is an important quantity for the evaluation of the stability properties. It represents the ability of the floating structure to absorb roll energy imparted to it by winds, waves or any other external effect.

### 2.3.8 Eccentric Loading

When a load is added at an arbitrarily place on a floating structure, its static response can be split into two components: It will sink deeper parallel to the original water plane and it will rotate (heel and/or trim) as well.

The parallel sinkage is caused by an increase of the total mass of the structure which requires a corresponding increase in displacement.

Heel or trim after a parallel sinkage will be caused by the generation of a moment. This moment is caused by a shift of the center of gravity and of the center of buoyancy. Both, they have been displaced from their originally vertical colinear positions.

Now, suppose a floating structure initially has a mass $m$, a displacement volume $\nabla_0$, a center of gravity $G_0$ and a center of buoyancy $B_0$, as given in figure 2.16. Then a mass $p$, placed on this structure, will result in a new volume of displacement $\nabla = \nabla_0 + \Delta \nabla_0$, a new center of gravity $G$ and a new center of buoyancy $B$.

No moment will be generated when the centers of gravity and buoyancy shift horizontally over equal distances. Using the first moment of masses, it can be seen that this is true whenever the mass $p$ has been placed above, at or under the center of the added displacement $\Delta \nabla_0$ caused by parallel sinkage.

![Figure 2.16: Parallel Sinkage](image)
CHAPTER 2. HYDROSTATICS

The first moments of the masses of the structure and the masses of displacement are defined here with respect to a line through \( G_0 \) and \( B_0 \):

\[
\begin{align*}
\{m + p\} \cdot a &= \{m\} \cdot 0 + \{p\} \cdot c \\
\{\rho \nabla_0 + \rho \cdot \Delta \nabla_0\} \cdot b &= \{\rho \nabla_0\} \cdot 0 + \{\rho \cdot \Delta \nabla_0\} \cdot c \\
\text{new} &= \text{old} + \text{change}
\end{align*}
\]

(2.42)

Because \( p = \rho \cdot \Delta \nabla_0 \), it follows that \( a = b \). This means that the new center of buoyancy \( B \) and the new center of gravity \( G \) are situated on one vertical line; no additional heeling moment remains.

The equilibrium after placing a load at an arbitrarily place on the structure, can be determined in two steps:

1. Place the mass \( p \) in the required horizontal plane vertically above or under the increased volume due to parallel sinkage, \( \Delta T_0 \), of the structure. When the water plane does not change very much during this draft increase, one can write:

\[
\Delta T_0 = \frac{p}{\rho \cdot A_{WL}}
\]

The center of this added displacement lies a distance \( \Delta T_0/2 \) directly above this original water plane. The center of buoyancy \( B_0 \) shifts to \( B \) because of the added volume of displacement \( \Delta \nabla_0 = p/\rho = \Delta T_0 \cdot A_{WL} \). The center of gravity shifts from \( G_0 \) to \( G \) because of the increase of the mass of the structure with \( p = \rho \cdot \Delta \nabla_0 \). These two shifts can be calculated easily with the first moments of masses and volumes, especially when the shape of the water plane does not change very much during the draft increase \( \Delta T_0 \):

\[
\begin{align*}
\text{horizontal part of } G_0G &= \frac{a \cdot m}{m + p} \\
\text{horizontal part of } B_0B &= \frac{a \cdot \nabla_0}{\nabla_0 + \Delta \nabla_0} = \frac{a \cdot m}{m + p}
\end{align*}
\]

(2.43)

Figure 2.17: Load Above Waterline Centre
2.3. STATIC FLOATING STABILITY

See figure 2.17 for the meaning of the symbols used here. When the structure hull is symmetric with respect to a plane through the points \( G_0, B_0 \) and \( M_0 \), then the center of the water plane lies in that symmetry plane, too. Then, we can place \( p \) above both, the center of buoyancy and the center of gravity. Because \( a = 0 \), both the horizontal part of \( G_0G \) and the horizontal part of \( B_0B \) are zero. The metacenters are given by:

\[
\frac{B_0M_0}{V_0} = \frac{I_T}{V_0} \quad \text{and} \quad BM = \frac{I_T}{V_0 + \Delta V_0} \tag{2.44}
\]

Finally, the vertical shifts of \( G_0 \) and \( B_0 \) are given by:

\[
\text{vertical part of } \frac{G_0G}{m+p} = \frac{(GO + b) \cdot m}{m+p} \\
\text{vertical part of } \frac{B_0B}{V_0 + \Delta V_0} = \frac{(BO + \Delta T_0/2) \cdot V_0}{m+p} \tag{2.45}
\]

2. Replace the resulting moment, caused by a movement of the mass \( p \) horizontally over a distance \( c \) to its desired position, by a heeling moment \( M_H \); see figure 2.18. The displacement mass, \( \rho V \), includes the mass \( p \):

\[
\nabla = \nabla_0 + \Delta \nabla_0 = \nabla_0 + \frac{p}{\rho} \tag{2.46}
\]

The heeling moment \( M_H \) must be equal to the righting stability moment \( M_S \):

\[
M_H = M_S \\
g \cdot p \cdot c \cdot \cos \phi = g \cdot \rho \nabla \cdot GN_\phi \cdot \sin \phi \tag{2.47}
\]

The heel angle, \( \phi \), follows from this moment equilibrium:

\[
\phi = \arccos \left( \frac{\rho \nabla \cdot GN_\phi \cdot \sin \phi}{p \cdot c} \right) \tag{2.48}
\]

If the static stability curve lever arms \( GZ = GN_\phi \cdot \sin \phi \) are known as functions of \( \phi \), then \( \phi \) can be determined iteratively.
For wall-sided structures one can substitute equation 2.35 in equation 2.48 and get:

\[
\phi = \arccos \left\{ \frac{\rho \nabla \cdot \{ GM + \frac{1}{2}BM \cdot \tan^2 \phi \}}{p \cdot c} \cdot \sin \phi \right\}
\]  
(2.49)

or using \(\tan \phi\):

\[
\frac{1}{2}BM \cdot \tan^3 \phi + GM \cdot \tan \phi = \frac{p \cdot c}{\rho \nabla}
\]  
(2.50)

This third degree equation in \(\tan \phi\) can be solved iteratively using the Regula-Falsi method by calculating the Left Hand Side (LHS) of the equation as a function of \(\phi\) until a value equal to the value of the Right Hand Side (RHS) has been found.

For small angles of heel and when the shape of the water plane does not change very much, the expression for \(\phi\) can be simplified to become:

\[
\phi = \arctan \left\{ \frac{p \cdot c}{\rho \nabla \cdot GM} \right\}
\]  
(2.51)

so that \(\phi\) can be found directly in this situation.

### 2.3.9 Inclining Experiment

Much of the data used in stability calculations depends only on the geometry of the structure. The total mass of the structure \(m\) follows from the displacement volume, \(\nabla\). The longitudinal position of the center of gravity, \(G\), follows simply from the longitudinal position of the center of buoyancy, \(B\), which can be determined from the under water geometry of the structure. The vertical position of the center of gravity must be known before the stability can be completely assessed for a given loading condition. Since the vertical position of the center of gravity, \(KG\), sometimes can be 10 times greater than the initial metacentric height \(GM\), \(KG\) must be known very accurately if the metacentric height is to be assessed with reasonable accuracy. \(KG\) can be calculated for a variety of loading conditions, provided it is accurately known for one precisely specified loading condition.

The displacement volume, \(\nabla\) can be calculated, given the geometry of the structure, in combination with measured drafts fore and aft (at both ends) in the upright condition \((\phi = \phi_0 \approx 0^\circ)\). With this displacement and the measured angle of heel \((\phi = \phi_1)\) after shifting a known mass \(p\) on the structure over a distance \(c\), the vertical position of the center of gravity can be found easily by writing equation 2.51 in another form:

\[
GM = \frac{p \cdot c}{\rho \nabla \cdot \tan (\phi_1 - \phi_0)} \text{ with: } \phi_0 \ll \phi_1 \text{ and } \phi_1 < \pm 10^\circ
\]  
(2.52)

The known underwater geometry of the structure and this "measured" \(GM\) value yields the vertical position of the center of gravity \(G\).

An experiment to determine the metacentric height in this way is called an inclining experiment. The purposes of this experiment are to determine the displacement, \(\nabla\), and the position of the center of gravity of the structure under a precisely known set of conditions. It is usually carried out when construction is being completed (afloat) before leaving the construction yard. An additional inclining experiment may be carried out following an extensive modernization or a major refit. Keep in mind that \(\nabla\) and \(G\) have to be corrected when the mass \(p\) is removed from the structure, again.
2.3.10 Free Surface Correction

Free surfaces of liquids inside a floating structure can have a large influence on its static stability; they reduce the righting moment or stability lever arm.

Figure 2.19: Floating Structure with a Tank Partially Filled with Liquid

When the structure in figure 2.19 heels as a result of an external moment $M_H$, the surface of the fluid in the tank remains horizontal. This means that this free surface heels relative to the structure itself, so that the center of gravity of the structure (including liquid) shifts. This shift is analogous to the principle of the shift of the center of buoyancy due to the emerged and immersed wedges, as discussed above. Of course, the under water geometry of the entire structure and the boundaries of the wedges at the water plane as well as in the tank play a role. They determine, together with the angle of heel, the amount by which the centers of buoyancy and of gravity now shift.

In case of a vertical wedge boundary, (a wall-sided tank) as given in figure 2.20, this shift can be calculated easily with the first moment of volumes with triangular cross sections. In this figure, $\rho'$ is the density of the liquid in the tank, $v$ is the volume of the liquid in the tank and $i$ is the transverse moment of inertia (second moment of areas) of the surface of the tank.

Figure 2.20: Effect of a Liquid Free Surface in a Wall Sided Tank
the fluid in the tank ($\phi = 0$). The center of gravities are marked here by $b$ and not by $g$, to avoid confusions with the acceleration of gravity.

The influence of the movement of the mass $\rho'v$ from $b$ to $b_\phi$ on the position of the overall center of buoyancy can also be determined using the first moment of volumes with triangular cross sections:

<table>
<thead>
<tr>
<th>direction</th>
<th>shift of center of gravity of fluid</th>
<th>shift of center of gravity of total structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>horizontal</td>
<td>$\frac{1}{v} \cdot \tan \phi$</td>
<td>$\frac{\rho' l}{\rho V} \cdot \tan \phi$</td>
</tr>
<tr>
<td>vertical</td>
<td>$\frac{1}{v} \cdot \frac{1}{2} \tan^2 \phi$</td>
<td>$\frac{\rho' l}{\rho V} \cdot \frac{1}{2} \tan^2 \phi$</td>
</tr>
</tbody>
</table>

The amount of fluid in the tank, $v$, does not matters, only the transverse moment of inertia of the surface of the fluid in the tank, $i$, counts. This has been made more clear in figure 2.21.

Figure 2.21: Metacentric Height Reduction Caused by Free Surfaces in Wall Sided Tanks

This means that the righting stability lever arm will be reduced by $\overline{GG''} \cdot \sin \phi$, with:

$$\overline{GG''} = \frac{\rho' l}{\rho V} \cdot \left(1 + \frac{1}{2} \tan^2 \phi\right)$$  \hspace{1cm} (2.53)
The magnitude $\overline{G\sigma''}$ is called the **free surface correction** or the **reduction of the metacentric height**.

For small angles of heel for which $\frac{1}{2}\tan^2 \phi$ is small relative to 1.0, one can write for the reduction of the metacentric height:

$$\overline{G\sigma''} \approx \overline{G\sigma} = \frac{\rho' i}{\rho V} \quad (2.54)$$

For more free surfaces (more tanks) the reduction of the metacentric height becomes:

$$\overline{G\sigma''} = \sum \frac{\rho' i}{\rho V} \cdot \left(1 + \frac{1}{2} \tan^2 \phi\right) \quad (2.55)$$

**Figure 2.22:** $GZ$-Curve, Corrected for Free Surface

Note that the effect of the free surface on the stability lever arm (see figure 2.22) is independent of the position of the tank in the floating structure; the tank can be at any height in the structure and at any position in its length or breadth.

The effect is also independent of the amount of liquid in the tank, provided that the moment of inertia (second moment of areas) of the free surface remains substantially unchanged when inclined. The moment of inertia of the free surface of a tank which is almost empty or almost full can change significantly during an inclining experiment. This is why tanks which cannot be completely full or empty during an inclining experiment are specified to be half full.

It is common practice to treat the free surface effect as being equivalent to a virtual rise of the center of gravity of the structure. One should appreciate that this is merely a convention which has no factual basis in reality.

A similar free surface effect can arise from movements of granular cargo such as grain which is stowed in bulk. Here, however, there is no simple relation between the angle of inclination of the ship and the slope of the surface of the grain. At small angles of heel the grain is not likely to shift, so there is no influence on initial stability. When the heel
angle becomes great enough, a significant shift can take place very abruptly; its effect can be equally abrupt!
Chapter 3

CONSTANT POTENTIAL FLOW PHENOMENA

3.1 Introduction

This chapter introduces hydrodynamics by treating the simplest of flows: constant, incompressible, potential flows; time is therefore unimportant. These flows obey reasonably simplified laws of fluid mechanics and mathematics as will be explained in this chapter. In spite of its limitations, potential flow is widely used in fluid mechanics. It often gives at least a qualitative impression of flow phenomena. Discrepancies between real and idealized flows are often accommodated via experimentally determined coefficients.

The concepts introduced in this chapter will be extended and applied in later chapters for a wide variety of solutions including those involving waves. Indeed, potential theory can very easily be extended to time dependent flows as well. This chapter starts by defining a potential flow field and its properties in a very general way. The power of the method will become more obvious as it is applied to constant flow situations later in this chapter. Much of the information in this chapter may be well-known to the reader; it can be seen as a quick review. In any case, this chapter introduces the notation used in many of the remaining chapters as well.

3.2 Basis Flow Properties

All flows are treated here as being non-viscous, incompressible, continuous and homogeneous. There are no cavitation holes or gas bubbles in the fluid.

3.2.1 Continuity Condition

Consider first a differential element of fluid within a larger body of a compressible flow. This block has edges with dimensions $dx$ by $dy$ by $dz$ oriented parallel to the axes of a right-handed $(x, y, z)$-axis system as shown in figure 3.1 (only the $y$-component of velocity is shown).

In the most general situation, the increase of mass per unit of time of the fluid with density $\rho$ in this block is given by:

$$\frac{\partial m}{\partial t} = -\frac{\partial}{\partial t} (\rho \cdot dx \, dy \, dz) = -\frac{\partial \rho}{\partial t} \cdot dx \, dy \, dz$$ \hspace{1cm} (3.1)$$

The minus sign comes from the fact that if the mass transport increases across the block, then there will be a net loss of mass from within it. The continuous and homogeneous flow will have a velocity, $\vec{V}$, defined in terms of velocity components, $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$ along the three axes, respectively. This is here still a compressible flow, so the density of the fluid in the flow is a function of $x$, $y$ and $z$; $\rho(x, y, z)$. Note that only a flow component perpendicular to a face of the block will cause a mass transport in (or out) through that face of the block.

Looking first along the $x$-axis, the mass flow through plane $dy \, dz$ during a unit of time $dt$ into the block at $x$ is:

$$m_{\text{in along } x\text{-axis}} = \rho \cdot u \cdot dy \, dz \, dt$$

and the mass flow through plane $dy \, dz$ during a unit of time $dt$ out of the block at $x + dx$ is:

$$m_{\text{out along } x\text{-axis}} = \left[ \rho \cdot u + \frac{\partial (\rho \cdot u)}{\partial x} \cdot dx \right] \cdot dy \, dz \, dt$$

Then the net mass flow through the block per unit of time is the mass flow out per unit of time at $x + dx$ minus the mass flow in per unit of time at $x$:

$$\left( \frac{\partial m}{\partial t} \right)_{\text{along } x\text{-axis}} = \left[ \rho \cdot u + \frac{\partial (\rho \cdot u)}{\partial x} \cdot dx \right] \cdot dy \, dz - \rho \cdot u \cdot dy \, dz$$ \hspace{1cm} (3.2)$$

$$= \frac{\partial (\rho \cdot u)}{\partial x} \cdot dx \, dy \, dz$$ \hspace{1cm} (3.3)$$
3.2. BASIS FLOW PROPERTIES

Similarly, along the $y$ and $z$ directions:

\[
\left( \frac{\partial m}{\partial t} \right)_{\text{along } y\text{-axis}} = \left[ \rho v + \frac{\partial (\rho v)}{\partial y} \cdot dy \right] \cdot dx \cdot dz - \rho v \cdot dx \cdot dz
\]

\[
= \frac{\partial (\rho v)}{\partial y} \cdot dx \cdot dy \cdot dz
\]

\[ (3.4) \]

\[
\left( \frac{\partial m}{\partial t} \right)_{\text{along } z\text{-axis}} = \left[ \rho w + \frac{\partial (\rho w)}{\partial z} \cdot dz \right] \cdot dx \cdot dy - \rho w \cdot dx \cdot dy
\]

\[
= \frac{\partial (\rho w)}{\partial z} \cdot dx \cdot dy \cdot dz
\]

\[ (3.5) \]

Combining equations 3.1 through 3.5 provides the **Continuity Equation**:

\[
\frac{\partial m}{\partial t} = -\frac{\partial \rho}{\partial t} \cdot dx \cdot dy \cdot dz = \left\{ \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} \right\} \cdot dx \cdot dy \cdot dz
\]

\[ (3.6) \]

or:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0
\]

\[ (3.7) \]

In a more elaborate vector notation this equation becomes:

\[
\nabla \cdot (\rho \cdot \vec{V}) = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \cdot \vec{V}) = 0
\]

\[ (3.8) \]

In the above equation, $\nabla$ or $\nabla$ (nabla) denotes the sum of the partial derivatives:

\[
\nabla = \nabla (\text{nabla}) = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}
\]

\[ (3.9) \]

Note that nabla will refer to the displacement volume of a floating body in chapter 2 and later on; this is an entirely different quantity!

Now, the fluid will be treated as incompressible; the density $\rho$ is considered to be constant. This can be done in many cases in hydrodynamics. This allows equation 3.7 to be simplified by cancelling $\rho$:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

\[ (3.10) \]

or in a more elaborate vector notation:

\[
\nabla \cdot \vec{V} = 0 \quad \text{or} \quad \nabla \cdot \vec{V} = 0
\]

\[ (3.11) \]

If one is working with only two dimensions, then the third dimension above (usually $z$) can be neglected; all terms in that direction are set to zero.

It is left to the reader to verify that in a two-dimensional homogeneous incompressible flow the continuity condition in equation 3.10 can be written in polar coordinates (see figure 3.2) as:

\[
\frac{1}{r} \left( \frac{\partial (r \cdot v_r)}{\partial r} - \frac{\partial v_\theta}{\partial \theta} \right) = 0 \quad \text{or} \quad \frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} = 0
\]

\[ (3.12) \]
CHAPTER 3. CONSTANT POTENTIAL FLOW PHENOMENA

Another very simple form of the continuity equation is:

\[ Q = A \cdot V \] (3.13)

in which \( Q \) is the volume flow rate \((m^3/s)\), \( A \) is the cross section area perpendicular to the velocity \((m^2)\) and \( V \) is the velocity component perpendicular to the area \((m/s)\) averaged over the cross section area.

### 3.2.2 Deformation and Rotation

The stresses in the fluid will now be related to deformations and rotations of the fluid particle. The cube, considered before, will be deformed and rotated by forces on its sides. Mathematically, the velocity components \( u(x, y, z, t) \), \( v(x, y, z, t) \) and \( w(x, y, z, t) \) can be expanded in a Taylor series. That series for \( u(x, y, z, t) \), for example, would become:

\[
u + \Delta u = u + \frac{\partial u}{\partial x} \cdot \Delta x + \frac{\partial u}{\partial y} \cdot \Delta y + \frac{\partial u}{\partial z} \cdot \Delta z + \frac{\partial u}{\partial t} \cdot \Delta t + \ldots \text{ higher order terms...} \quad (3.14)
\]

The higher order terms can be ignored as \( \Delta u \to 0 \) and \( u \) can be cancelled in both sides of the equation. Then equation 3.14 reduces to:

\[
du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy + \frac{\partial u}{\partial z} \cdot dz + \frac{\partial u}{\partial t} \cdot dt \quad (3.15)
\]

which can - after ordering terms - be written as:

\[
du = \frac{\partial u}{\partial x} \cdot dx + \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \cdot dy + \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \cdot dy + \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \cdot dz + \frac{1}{2} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \cdot dz + \frac{\partial u}{\partial t} \cdot dt \quad (3.16)
\]

= stretch + dilatation + rotation + translation

For simplicity only two dimensions will be considered now, a cross section of the fluid particle in the \((x, y)\)-plane as shown in figure 3.3. The velocity components in the other planes can be found in a similar way.
3.2. BASIS FLOW PROPERTIES

Figure 3.3: Deformation and Rotation

2-D Deformation

From figure 3.3 follows the deformation:

\[ \frac{\partial v}{\partial x} = \tan \hat{\alpha} \approx \hat{\alpha} \quad \text{and} \quad \frac{\partial u}{\partial y} = \tan \hat{\beta} \approx \hat{\beta} \]  \hspace{1cm} (3.17)

2-D Dilatation

The deformation velocity (or dilatation) becomes:

\[ \frac{\hat{\alpha} + \hat{\beta}}{2} = \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \]  \hspace{1cm} (3.18)

2-D Rotation

In case of a small square surface element \((dx = dy)\), the diagonal rotates over an angle:

\[ \varphi = \frac{1}{2} \left( \hat{\alpha} - \hat{\beta} \right) \quad \text{because:} \quad \frac{\pi}{4} - \hat{\alpha} + \varphi = \frac{\pi}{4} - \hat{\beta} - \varphi \]  \hspace{1cm} (3.19)

Thus, the angular velocity (or rotation) of the diagonal is:

\[ \dot{\varphi} = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \]  \hspace{1cm} (3.20)

which is zero by definition in potential theory, as will be treated in the next section.
3.3 Potential Flow Concepts

A velocity potential of a flow is simply a mathematical expression, which has the property that the velocity component in a point in the fluid in any chosen direction is simply the derivative of this potential function in that point to that chosen direction. This will be explained in more detail in the next sections.

3.3.1 Potentials

If $u$, $v$ and $w$ are velocity components along the $x$-, $y$- and $z$-axes respectively, the increase of the so-called potential value between two points $A$ and $B$ in the fluid - see figure 3.4 - is defined as:

$$
\Delta \Phi_{A\to B} = \int_A^B \nabla \cdot d\vec{s} = \int_A^B (u \cdot dx + v \cdot dy + w \cdot dz)
$$

$$
= \int_A^B \left( \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz \right) = \int_A^B d\Phi = \Phi(B) - \Phi(A) \tag{3.21}
$$

In fact, this follows from the definition of the potential value itself.

![Figure 3.4: Definition of Velocity Potential](image)

This means that the increase of the potential value from $A$ to $B$ is independent of the chosen integration route between those points.

Potential Lines

Equation 3.21 also means that potential lines are curves defined in such a way that:

$$
\Phi = \text{constant} \tag{3.22}
$$

Using the definitions above, this implies in 3-D rectangular coordinates that:

$$
d\Phi = u \, dx + v \, dy + w \, dz \tag{3.23}
$$

or in 2-D polar coordinates:

$$
d\Phi = v_r \, dr + r \, v_\theta \, d\theta \tag{3.24}
$$
### Potential Function

Equation 3.23 includes a very convenient definition of the potential function: A potential function, \( \Phi \), associated with a potential flow field is a mathematical expression having the convenient property that at any point in the flow, the velocity component in any chosen direction is simply the derivative of this potential function in that chosen direction:

\[
\begin{align*}
  u &= \frac{\partial \Phi}{\partial x} \\
  v &= \frac{\partial \Phi}{\partial y} \\
  w &= \frac{\partial \Phi}{\partial z}
\end{align*}
\]

(3.25)

While potential functions can be defined (in theory) in any number of dimensions, it is easiest to understand them in a two-dimensional situation; this will be used here. In 2-D polar coordinates - see figure 3.2 - the potential functions are given by:

\[
\begin{align*}
  v_r &= \frac{\partial \Phi}{\partial r} \\
  v_\theta &= \frac{1}{r} \cdot \frac{\partial \Phi}{\partial \theta}
\end{align*}
\]

(3.26)

All potential theory solutions must fulfill the following two conditions:

1. **Laplace Equation**

   Applying the continuity condition (equation 3.10) to the velocities in equation 3.25 provides the Laplace equation for an incompressible fluid:

   \[
   \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0
   \]

   or:

   \[
   \nabla^2 \Phi = 0
   \]

   (3.27)

2. **Rotation Free**

   This condition - see also equations 3.16 and 3.20 - follows from the potential theory itself. One can write for the \((x, y)\)-plane:

   \[
   \begin{align*}
   u &= \frac{\partial \Phi}{\partial x} \quad \text{so:} \quad \frac{\partial u}{\partial y} &= \frac{\partial}{\partial y} \left( \frac{\partial \Phi}{\partial x} \right) = \frac{\partial^2 \Phi}{\partial y \partial x} \\
   v &= \frac{\partial \Phi}{\partial y} \quad \text{so:} \quad \frac{\partial v}{\partial x} &= \frac{\partial}{\partial x} \left( \frac{\partial \Phi}{\partial y} \right) = \frac{\partial^2 \Phi}{\partial x \partial y}
   \end{align*}
   \]

   (3.28)

   Because:

   \[
   \frac{\partial^2 \Phi}{\partial y \partial x} = \frac{\partial^2 \Phi}{\partial x \partial y}
   \]

   one can write:

   \[
   \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = 0
   \]

   (3.29)

   This can be applied similarly to the other planes as well so that:

   \[
   \begin{align*}
   \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 0 \quad \text{in the} \ (x, y)\text{-plane} \\
   \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} &= 0 \quad \text{in the} \ (y, z)\text{-plane} \\
   \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} &= 0 \quad \text{in the} \ (x, z)\text{-plane}
   \end{align*}
   \]

   (3.30)

   or in 2-D polar coordinates:

   \[
   \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r} \cdot \frac{\partial v_r}{\partial \theta} = 0
   \]

   (3.31)
In vector notation, the velocity can be written in component form in a rectangular coordinate system as:

\[ \vec{V}(x, y, z) = u(x, y, z) \vec{x} + v(x, y, z) \vec{y} + w(x, y, z) \vec{z} \]  

(3.32)

or in 2-D polar coordinates - see figure 3.2 with \( \vec{x}, \vec{y}, \vec{e}_r, \) and \( \vec{e}_\theta \) as unit vectors - as:

\[ \vec{V}(r, \theta) = v_r \vec{e}_r + v_\theta \vec{e}_\theta \]  

(3.33)

Because \( \partial v/\partial x, \partial u/\partial y \) and \( \partial w/\partial z \) are the three rotation components in a Cartesian axes system, equations 3.30 mean that any potential fluid flow is free of rotation. This condition can be tested easily in a real flow with a free surface. Sprinkle some short wooden toothpicks on the flow. If these do not turn as they pass a body in this flow, then (that portion of) the flow is rotation-free. Of course, if these floats come very close to the body they will turn in the (viscous) boundary layer near the body; this potential flow condition is not fulfilled there.

### 3.3.2 Euler Equations

In 1755, Leonard Euler applied Newton’s second law (force = mass x acceleration) to non-viscous and incompressible fluids. Doing this in the \( x \)-direction for a fluid mass element \( dm \) - as given in figure 3.1 - yields:

\[ dm \cdot \frac{Du}{Dt} = \rho \cdot dx dy dz \cdot \frac{Du}{Dt} = - \frac{\partial p}{\partial x} dx dy dz \]  

(3.34)

so that the instationary Euler equation in the \( x \)-direction becomes:

\[ \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt} \]

\[ = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \]

\[ = - \frac{1}{\rho} \frac{\partial p}{\partial x} \quad \text{with:} \quad u = u(x, y, z, t) \]  

(3.35)

In the same way, one can write the instationary Euler equations in the other two directions:

\[ \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \frac{dx}{dt} + \frac{\partial v}{\partial y} \frac{dy}{dt} + \frac{\partial v}{\partial z} \frac{dz}{dt} \]

\[ = \frac{\partial v}{\partial t} + w \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \]

\[ = - \frac{1}{\rho} \frac{\partial p}{\partial y} \quad \text{with:} \quad v = v(x, y, z, t) \]  

(3.36)

\[ \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \]

\[ = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \]

\[ = - \frac{1}{\rho} \frac{\partial p}{\partial z} \quad \text{with:} \quad w = w(x, y, z, t) \]  

(3.37)
Thus, the Euler equations for a non-viscous and incompressible flow are given by:

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial z}
\end{align*}
\]

in x-direction

in y-direction

in z-direction (3.38)

### 3.3.3 Bernoulli Equation

A flow can have kinetic, pressure and potential energy. Since there is no friction within the potential flows being discussed in this chapter, energy is conserved along a stream line. In 1738, Daniel Bernoulli, one of several members of this illustrious Swiss family - published treatises on fluid flow. This led to the development of an equation for the energy relation which bears his name; it is valid along any stream line.

#### Instationary or Unsteady Flow

All properties of an instationary flow vary (at all locations) as a function of time. The velocity terms in the Euler equations 3.38 can be written in terms of the velocity potential \( \Phi \) by:

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{\partial \Phi}{\partial x} \cdot \frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial \Phi}{\partial x} \right)^2 \\
\frac{\partial v}{\partial y} &= \frac{\partial \Phi}{\partial y} \cdot \frac{\partial^2 \Phi}{\partial x \partial y} = \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial \Phi}{\partial y} \right)^2 \\
\frac{\partial w}{\partial z} &= \frac{\partial \Phi}{\partial z} \cdot \frac{\partial^2 \Phi}{\partial x \partial z} = \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial \Phi}{\partial z} \right)^2 
\end{align*}
\]

etc., etc.

Substituting this in equations 3.38 provides:

\[
\begin{align*}
\frac{\partial}{\partial x} \left\{ \frac{\partial \Phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial y} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 \right] + \frac{p}{\rho} \right\} &= 0 \\
\frac{\partial}{\partial y} \left\{ \frac{\partial \Phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial y} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 \right] + \frac{p}{\rho} \right\} &= 0 \\
\frac{\partial}{\partial z} \left\{ \frac{\partial \Phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial y} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 \right] + \frac{p}{\rho} \right\} &= 0
\end{align*}
\]

(3.40)

Differentiation of the expressions between braces \( \{\ldots\} \) with respect to \( x, y \) or \( z \) gives zero. So these expressions are functions of only time, \( C(t) \), which provides the Bernoulli equation for an instationary flow:
\[
\frac{\partial \Phi}{\partial t} + \frac{1}{2}V^2 + \frac{p}{\rho} + g\zeta = C(t) \tag{3.41}
\]

in which:

\[V^2 = u^2 + v^2 + w^2 = \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial y} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 \tag{3.42}\]

**Stationary or Steady Flow**

In a stationary flow the derivative of the potential, \(\frac{\partial \Phi}{\partial t}\), becomes zero and the function of time, \(C(t)\), becomes a constant, \(C\); all time functions are now absent. In this case, the Bernoulli equation becomes:

\[\frac{1}{2}V^2 + \frac{p}{\rho} + g\zeta = C \quad \text{(along a stream line)} \tag{3.43}\]

Often, the Bernoulli equation for a stationary flow is written in other forms:

\[\frac{V^2}{2g} + \frac{p}{\rho g} + z = \text{Constant} \quad \text{(a)} \quad \text{or} \quad \frac{1}{2}\rho V^2 + p + \rho g z = \text{Constant} \quad \text{(b)} \tag{3.44}\]

The Bernoulli equation for stationary flow in 3.44(a) results in units of length (often called head of fluid) and is often favoured by civil engineers, whereas the form of equation 3.44(b) results in pressure and is often favoured by mechanical engineers and naval architects.

### 3.3.4 2-D Streams

Physically, a **streamline** is a 'line which follows the flow'; if a thin stream of dye were to be added to the flow, it would follow and physically define a streamline.
3.3. POTENTIAL FLOW CONCEPTS

The vector \( \vec{n} \) in figure 3.5 is the normal vector on an arbitrary line through a point \( A \) on the left streamline and a point \( B \) on the right streamline. The rate of flow between two streamlines in the \((x, y)\)-plane (see figures 3.5-a and 3.5-b) is given by:

\[
\Delta \Psi_{A-B} = \int_{A}^{B} (\vec{V} \cdot \vec{n}) \cdot ds = \int_{A}^{B} (u \cdot dy - v \cdot dx)
\]

\[
= \int_{A}^{B} \left( \frac{\partial \Psi}{\partial y} dy + \frac{\partial \Psi}{\partial x} dx \right) = \int_{A}^{B} d\Psi = \Psi(B) - \Psi(A) \tag{3.45}
\]

in which \( \Delta \Psi_{A-B} \) is the volume of flow per unit of time between two streamlines through \( A \) and \( B \).

2-D Stream Lines

This rate of flow remains constant when \( A \) and \( B \) follow their streamlines. The volume of flow between any two streamlines is therefore directly proportional to the difference between the - to be defined - stream function values of those two streamlines. Furthermore, the velocity must increase as the streamlines come closer together and decrease as they diverge while following a stream path. This follows from continuity of the incompressible flow. Mathematically, a streamline is simply a curve for which:

\[
\Psi = \text{constant} \tag{3.46}
\]

Since no flow takes place across streamlines, then in 2-D rectangular coordinates:

\[
d\Psi = u \ dy - v \ dx \tag{3.47}
\]

or in 2-D polar coordinates:

\[
d\Psi = v_r \ r \ d\theta - v_\theta \ dr \tag{3.48}
\]

Just like the potential function has a stream function no direct physical meaning; its derivatives however are very useful.

2-D Stream Function

Much like the potential function defined above, the stream function, \( \Psi \), is a mathematical expression such that:

\[
\boxed{u = \frac{\partial \Psi}{\partial y}} \hspace{1cm} \text{(a)} \quad \text{and} \quad \boxed{v = -\frac{\partial \Psi}{\partial x}} \hspace{1cm} \text{(b)} \tag{3.49}
\]

or in polar coordinates:

\[
\boxed{v_r = \frac{1}{r} \ \frac{\partial \Psi}{\partial \theta}} \hspace{1cm} \text{(a)} \quad \text{and} \quad \boxed{v_\theta = -\frac{\partial \Psi}{\partial r}} \hspace{1cm} \text{(b)} \tag{3.50}
\]
3.3.5 Properties

A few basic additional principles are reviewed in this section before continuing with the more classical potential flow mechanics and mathematics.

Orthogonality

The velocity components, \( u \) and \( v \), can be expressed in two different but equivalent ways:

\[
\begin{align*}
\frac{\partial \Phi}{\partial x} &= + \frac{\partial \Psi}{\partial y} \quad \text{(a)} \quad \text{and} \quad \frac{\partial \Phi}{\partial y} = - \frac{\partial \Psi}{\partial x} \quad \text{(b)}
\end{align*}
\]  

(3.51)

Since \( \frac{\partial \Phi}{\partial y} \) is the negative reciprocal of \( \frac{\partial \Psi}{\partial x} \), the stream lines and potential lines must be orthogonal; the stream lines and potential lines always cross each other at right angles. This fact can be handy when sketching flow patterns using stream lines and potential lines. In 2-D polar coordinates, the velocity components, \( v_r \) and \( v_\theta \), are:

\[
\begin{align*}
\frac{v_r}{r} &= - \frac{\partial \Phi}{\partial r} = \frac{1}{r} \cdot \frac{\partial \Psi}{\partial \theta} \quad \text{(a)} \quad \text{and} \quad \frac{v_\theta}{r} = \frac{1}{r} \cdot \frac{\partial \Phi}{\partial \theta} = - \frac{\partial \Psi}{\partial r} \quad \text{(b)}
\end{align*}
\]  

(3.52)

Impervious Boundary

There is no flow across a streamline so that any impervious flow boundary, therefore, must also be a streamline. This means that:

\[
\begin{align*}
\frac{\partial \Phi}{\partial n} &= 0 \\
\Psi &= \text{constant}
\end{align*}
\]  

at any impervious boundary

(3.53)

in which \( n \) is the normal to the (impervious) surface. These are important relations which will be used repeatedly.

Conditions far from a Disturbance

It is usually assumed that nothing special happens as one moves far away from the area of interest. In mathematical terms, this means that:

\[
\text{if} \quad x \gg 0 \quad \text{and/or} \quad y \gg 0 \quad \text{then:} \quad \Psi \to \Psi_\infty \quad \text{and} \quad \Phi \to \Phi_\infty
\]  

(3.54)

The subscript \( \infty \) denotes a condition which exists far away from the location of primary interest.

Steady and Unsteady Flow

Steady flow means that at any chosen location - thus also at a point on any given streamline - the velocity is constant and independent of time. This means that:

\[
\frac{\partial \vec{V}}{\partial t} = 0 \quad \text{(steady flow)}
\]  

(3.55)

Obviously, unsteady flow is any flow condition which does not satisfy this relation. Waves are a good example of unsteady flow, by the way.
3.4. POTENTIAL FLOW ELEMENTS

Uniform Flows

One speaks of uniform flow when the velocity does not change as one progresses along a streamline at any instant in time. This means that:

$$\frac{\partial V}{\partial s} = 0 \quad \text{(uniform flow)}$$  \hspace{1cm} (3.56)

in which $s$ is a distance measured along the streamline.

3.4 Potential Flow Elements

Potential flows - described via convenient mathematical formulas - can also be superposed. This means that a series of simple flow situations can be added in order to produce a more complex flow. If these elements fulfil the Laplace equation, the sum of these elements does this too.

This section describes the basic building blocks used in this process; superposition comes later in this chapter.

3.4.1 Uniform Flow Elements

The simplest solution of the Laplace equation is:

$$\Phi = U \cdot x$$  \hspace{1cm} (3.57)

Its second derivative in the $x$-direction is zero and all derivatives in the $y$- and $z$-directions are also zero; so this potential satisfies the Laplace equation. The meaning of this potential
is a uniform steady flow in the $x$-direction, since the only velocity component is $u(x, y) = U$. This flow has a series of equally spaced parallel stream lines parallel to the $x$-axis, as shown in figure 3.6-a. Its stream function follows from:

$$u = \frac{\partial \Psi}{\partial y} = U \quad \text{so that} \quad \Psi = U \cdot y \quad (3.58)$$

The flow rate (in m$^3$/s for example - or actually in m$^2$/s because only a unit thickness in the $z$-direction is being considered) between any two stream lines separated by an amount $\Delta \Psi$ is constant and is proportional to $\Delta \Psi$ (or $\Delta y$ as well in this case). Thus for a uniform flow in the positive $x$-direction, the potential and stream functions are given by:

$$|\Phi = +U \cdot x| \quad (a) \quad \text{and} \quad |\Psi = +U \cdot y| \quad (b) \quad (3.59)$$

If the flow is in the negative $x$-direction (see figure 3.6-b), the potential and stream functions will change sign:

$$|\Phi = -U \cdot x| \quad (a) \quad \text{and} \quad |\Psi = -U \cdot y| \quad (b) \quad (3.60)$$

### 3.4.2 Source and Sink Elements

A **source** is a point from which flow radiates outward in all directions as shown in figure 3.6-c. Its location is often denoted by a small circle enclosing the letters SO or a plus sign. Its potential and stream functions are given most easily in polar coordinates:

$$\Phi = + \frac{Q}{2\pi} \cdot \ln r \quad (a) \quad \text{and} \quad \Psi = + \frac{Q}{2\pi} \cdot \theta \quad (b) \quad (3.61)$$

in which $Q$ is the strength of the source (or flow rate) with units of m$^2$/s for a two-dimensional flow.

It requires some algebra to show that this potential satisfies the Laplace equation. The potential lines are circles centered at the source location: $r =$ constant; its stream lines are radial lines: $\theta =$ constant.

The Cartesian expression of the potential and stream functions looks like:

$$\Phi = + \frac{Q}{2\pi} \cdot \ln \sqrt{x^2 + y^2} \quad (a) \quad \text{and} \quad \Psi = + \frac{Q}{2\pi} \cdot \arctan \left( \frac{y}{x} \right) \quad (b) \quad (3.62)$$

The velocity in the radial direction can be found from the derivative of the potential function with respect to $r$, so:

$$v_r = \frac{\partial \Phi}{\partial r} = \frac{1}{r} \cdot \frac{\partial \Psi}{\partial \theta} = \frac{Q}{2\pi r} \quad (3.63)$$

There is no flow in the tangential direction, because:

$$v_\theta = \frac{1}{r} \cdot \frac{\partial \Phi}{\partial \theta} = - \frac{\partial \Psi}{\partial r} = 0 \quad (3.64)$$

This flow is called a source flow because the amount of fluid passing through a circle at radius $r$ is always $Q$. The fluid is free of rotation, because the source strength, $Q$, is constant in the fluid. However, the origin, $r = \sqrt{x^2 + y^2} = 0$, is a singular point; the
velocities go to infinity at this point. Therefore, the potential theory is not valid there and this point has to be excluded when carrying out integrations in the fluid.

A **sink** is a negative source and thus has an inward radial flow. It is denoted by a circle enclosing the letters SK or a minus sign as shown in figure 3.6-d. In polar coordinates it can be expressed as:

\[
\Phi = -\frac{Q}{2\pi} \cdot \ln r \quad \text{and} \quad \Psi = -\frac{Q}{2\pi} \cdot \theta
\]  

(a) and

and in Cartesian coordinates as:

\[
\Phi = -\frac{Q}{2\pi} \cdot \ln \sqrt{x^2 + y^2} \quad \text{and} \quad \Psi = -\frac{Q}{2\pi} \cdot \arctan \left( \frac{y}{x} \right)
\]

(b) (3.66)

### 3.4.3 Circulation or Vortex Elements

A circulation or a vortex, \( \Gamma \), is a tangential flow (circular path) around a point as is shown in figure 3.6-e. The potential and stream functions are in this case:

\[
\Phi = +\frac{\Gamma}{2\pi} \cdot \theta \quad \text{and} \quad \Psi = -\frac{\Gamma}{2\pi} \cdot \ln r
\]

(a) and

or in Cartesian coordinates:

\[
\Phi = +\frac{\Gamma}{2\pi} \cdot \arctan \left( \frac{y}{x} \right) \quad \text{and} \quad \Psi = -\frac{\Gamma}{2\pi} \cdot \ln \sqrt{x^2 + y^2}
\]

(b) (3.68)

The velocities are now in a tangential **counter-clockwise direction**:

\[
v_\theta = \frac{1}{r} \cdot \frac{\partial \Phi}{\partial \theta} = -\frac{\partial \Psi}{\partial r} = \frac{\Gamma}{2\pi r}
\]

(3.69)

There is no velocity in radial direction, because:

\[
v_r = \frac{\partial \Phi}{\partial r} = \frac{1}{r} \cdot \frac{\partial \Psi}{\partial \theta} = 0
\]

(3.70)

One should remember that the circulation strength, \( \Gamma \), can be found from:

\[
\Gamma = \oint v_\theta \cdot ds = 2\pi r \cdot v_\theta = \text{constant}
\]

(3.71)

The fluid is free of rotation, because the circulation, \( \Gamma \), is constant in the fluid. However, the origin, \( r = \sqrt{x^2 + y^2} = 0 \), is a singular point; the velocities go to infinity at this point. Therefore, potential theory is not valid there and this point has to be excluded when carrying out integrations in the fluid.

The potential and stream functions of a circulation with tangential velocities in a **clockwise direction** are given by:

\[
\Phi = -\frac{\Gamma}{2\pi} \cdot \theta \quad \text{and} \quad \Psi = +\frac{\Gamma}{2\pi} \cdot \ln r
\]

(a) and

or in Cartesian coordinates by:
\[ \Phi = -\frac{\Gamma}{2\pi} \cdot \arctan \left( \frac{y}{x} \right) \quad \text{(a)} \quad \text{and} \quad \Psi = \frac{\Gamma}{2\pi} \cdot \ln \sqrt{x^2 + y^2} \quad \text{(b)} \quad (3.73) \]

Note that a circulation flow is a counterpart of a source or sink flow; \( \Phi \) and \( \Psi \) have similar forms but they are exchanged.

### 3.5 Superposition of Basic Elements

While each of the flow patterns from the four potential flow elements treated above can be quite easily visualized, they are not all that common, and a lot of interesting flow patterns are much more complex as well. The ’power’ of potential theory lies in the fact that because of its linearity superposition (or ’adding up’) of flow pattern components may be used to generate more complex flow patterns. This section concentrates on producing some of these more complex and interesting flow patterns.

#### 3.5.1 Methodology

Superposition may be carried out ’by hand’ using the following 5 steps; they are then most suited for simple cases. Superposition of only a few elements can be carried out by hand.

A. Place each potential flow element at its desired \( x, y \) location.

B. Draw the flow pattern from each element individually - without regard to the others.

C. Assign appropriate numerical values to each (independent) stream line.

D. The resulting stream function value at any point \( x, y \) can be found by adding the stream function values from each of the components at that point. (This is easiest at locations where streamlines from the various elements cross each other. Otherwise, a certain amount of interpolation may be needed to carry this out.) This yields a field of resulting stream function values, each associated with a particular location, \( x, y \).

E. Lines can now be sketched which connect locations having the same stream function values. This is much like drawing contour lines in map-making or a dot-to-dot drawing as in elementary school! Interpolation may again be necessary in this sketching process, however.

These steps are illustrated in figure 3.7 in which a uniform flow (from left to right) is combined with source flow. Parts A and B of the figure show half-planes of the two independent potential flow elements. These are superposed in part C of the figure thus completing the first three steps above. The results of the addition work are shown in part D; part E of the figure shows the resulting set of streamlines.

Whenever a larger number of elements are to be combined, the interpolation work involved in carrying out the above summations becomes too cumbersome for ’handwork’. A slightly different approach is most appropriate for computer processing.

Choose a single, \( X, Y \) coordinate system.
Figure 3.7: Superposition Steps Carried out by Hand
Place each of the potential flow elements at its desired \( X,Y \) location. Formulas are available to express each element’s stream and potential function in terms of its own (local) \( x,y \) coordinates.

Transform the coordinates for each flow element so that its contribution is expressed in terms of \( X \) and \( Y \) coordinates. These formulas will be used in step 5, below.

Choose a rectangular grid of points in the \( X,Y \) coordinate system at which the functions are to be evaluated. The area covered by these points should be the area for which the stream and potential lines are to be plotted.

A computer program - even a spreadsheet program will do - can be used to compute the relevant function value at each grid point. Even a modest, but modern PC can quickly generate values on a grid of 50 by 50 points.

The resulting data table can then be transferred to a 3-D plotting program to generate contour lines of stream or potential values as a function of \( X \) and \( Y \). These are the desired product.

Obviously, this method is not really hindered by the number of flow elements to be included; the formulas only become a bit longer in the program. Because the plotting program will interpolate anyway, it is no longer important that the computed grid point data have ‘convenient’ values.

The rest of this section details a number of simple and useful potential flow element combinations.

### 3.5.2 Sink (or Source) in Uniform Flow

This first example - still a bit simple - could represent the flow of oil to a well drilled into a reservoir through which the oil is flowing slowly. This situation has the opposite sense of the one used to illustrate the method, above; now, the flow pattern is determined by adding a sink to a uniform flow which goes from right to left. Using equations 3.52 and 3.66:

\[
\Psi = -\frac{Q}{2\pi} \cdot \arctan \left( \frac{y}{x} \right) - U_\infty \cdot y
\]

Figure 3.8 shows the curve separating the flow going to the sink from that passing (or ‘escaping’) the sink. All flow originating between the two asymptotes gets ‘caught’ by the sink.

Obviously, if the sink were to be replaced by a source, the flow coming from the source would then remain within the asymptotes as well.

### 3.5.3 Separated Source and Sink

Consider next a source and a sink with equal and opposite intensity, \( \frac{Q}{2\pi} \), separated by a distance of \( 2s \).

Once again, superposition is used to yield - using equations 3.66:

\[
\Psi_{\text{source}} = \frac{Q}{2\pi} \cdot \theta_1 = \frac{Q}{2\pi} \cdot \arctan \left( \frac{y}{x_1} \right) \quad \text{for the source} \quad (3.75)
\]

\[
\Psi_{\text{sink}} = -\frac{Q}{2\pi} \cdot \theta_2 = -\frac{Q}{2\pi} \cdot \arctan \left( \frac{y}{x_2} \right) \quad \text{for the sink} \quad (3.76)
\]
3.5. SUPERPOSITION OF BASIC ELEMENTS

The resulting stream lines are then (in polar coordinates for a change) lines with:

\[ \Psi = \frac{Q}{2\pi} \cdot \theta_1 - \frac{Q}{2\pi} \cdot \theta_2 = \text{constant} \]  

(3.77)

These stream lines are a set of circles, all centered on the \( y \) axis, and all passing through source and sink; see figure 3.9.

One can also write the resultant stream function in Cartesian coordinates as:

\[ \Psi = \frac{Q}{2\pi} \cdot \arctan \left( \frac{2}{x^2 + y^2 - s^2} \right) \]  

(3.78)
### 3.5.4 Source and Sink in Uniform Flow

The stream function is, in this case:

\[
\Psi = \frac{Q}{2\pi} \cdot \arctan \left( \frac{2y}{x^2 + y^2 - s^2} \right) + U_\infty y
\]  
(3.79)

The stream lines that one obtains now are shown in figure 3.10. Notice that the ellipse which surrounds the source and the sink (drawn a bit heavier in the figure) is a stream line; no flow takes place through that ellipse. The flow from source to sink stays inside; the constant current flow stays outside and passes around the form.

The physical interpretation of this is that one could obtain the same flow by replacing that ellipse with an impermeable object in a uniform flow.

![Figure 3.10: Source and Sink in Uniform Flow](image)

This approach can be extended even further as explained in the next section.

### 3.5.5 Rankine Ship Forms

The Englishman W.J.M. Rankine extended the above-mentioned approach about 1870. He did this by including additional, matched pairs of sources and sinks along the \(x\) axis in the uniform flow. They were always located symmetrically about the coordinate origin. By giving each matched source-sink pair its own spacing and strength (relative to the uniform flow), he was able to generate a fatter or thinner, or more blunt or pointed shape; it was always symmetric with respect to both the \(x\) and \(y\) axes, however. Use of relatively weaker source-sink pairs near the ends of the shape make it more 'pointed'. Since these forms can be made to somewhat resemble a horizontal slice of a ship (neglecting the fact that the stern has a bit different shape from the bow), they came to be known as **Rankine ship forms**. Flow computations with such forms are quite simple to carry out - and have been done for more than a century!

### 3.5.6 Doublet or Dipole

Suppose, in the discussion of a separate source and sink, above, one were to let the distance \(2s\) between them approach zero. This produces what is called a **doublet** or **dipole**.
3.5. SUPERPOSITION OF BASIC ELEMENTS

pole. The corresponding stream function is found by letting $s$ approach zero in equation 3.78:

$$
\Psi = \lim_{s \to 0} \left\{ \frac{Q}{2\pi} \cdot \arctan \left( \frac{2y s}{x^2 + y^2 - s^2} \right) \right\}
$$

As $x \to 0$ then $\arctan(x) \to x$, so that:

$$
\Psi = \lim_{s \to 0} \left\{ \frac{Q}{\pi s} \cdot \left( \frac{y}{x^2 + y^2 - s^2} \right) \right\}
$$

One must keep the term $\frac{Q}{\pi} s$ (that is renamed the doublet strength, $\mu$) constant and non-zero (otherwise there won’t be anything left; the origin is a singular point anyway!). On the other hand, $s^2$ becomes small even faster so that it can be neglected relative to $x^2 + y^2$. After a bit of algebra, the above stream function becomes:

$$
\Psi = \mu \cdot \frac{y}{x^2 + y^2} = \frac{\mu \cdot \sin \theta}{r}
$$

and similarly:

$$
\Phi = \mu \cdot \frac{x}{x^2 + y^2} = \frac{\mu \cdot \cos \theta}{r}
$$

In figure 3.11, both the stream (solid) and equipotential (dashed) lines are circles centered on the $y$ and $x$ axes respectively. All the circles pass through the origin, too.

![Figure 3.11: Doublet or Dipole Flow](image)

3.5.7 Doublet in Uniform Flow

The next step is to add a uniform flow to the doublet just discussed. If the flow goes along the negative $x$-axis, this yields:

$$
\Psi = \frac{\mu \cdot y}{x^2 + y^2} - U_\infty y
$$
and:

\[ \Phi = \frac{\mu \cdot x}{x^2 + y^2} - U_\infty \cdot x \]  \hspace{1cm} (3.85)

In polar coordinates this can be written as:

\[ \Psi = \frac{\mu \cdot \sin \theta}{r} - U_\infty \cdot r \cdot \sin \theta \]  \hspace{1cm} (3.86)

and:

\[ \Phi = \frac{\mu \cdot \cos \theta}{r} - U_\infty \cdot r \cdot \cos \theta \]  \hspace{1cm} (3.87)

It can be interesting to set \( \Psi = 0 \) in equation 3.84. This means that:

\[ \Psi = y \cdot \left[ \frac{\mu}{x^2 + y^2} - U_\infty \right] = \text{const.} = 0 \]  \hspace{1cm} (3.88)

This condition is true when:

1. \( y = 0 \)
   
   This means the \( x \) axis is a streamline. This can be expected because of the symmetry with respect to this axis.

2. \( \frac{\mu}{x^2 + y^2} - U_\infty = 0 \) \hspace{0.5cm} or, since \( x^2 + y^2 = R^2 \):
   \[ R^2 = \frac{\mu}{U_\infty} = \text{constant} \]
   
   This corresponds to a \textbf{circle}! Its radius is:

   \[ R = \sqrt{\frac{\mu}{U_\infty}} \text{ or } \mu = R^2 \cdot U_\infty \]  \hspace{1cm} (3.89)

The radius of the cylinder and the undisturbed current velocity, \( U_\infty \), thus, determines the necessary doublet strength value, \( \mu \).

Substituting this value for \( \mu \) then yields for the general equations in polar coordinates:

\[ \Psi = \frac{R^2 \cdot U_\infty \cdot \sin \theta}{r} - U_\infty \cdot r \cdot \sin \theta = R \cdot U_\infty \left( \frac{R}{r} - \frac{r}{R} \right) \cdot \sin \theta \]  \hspace{1cm} (3.90)

for the stream function and for the potential function:

\[ \Phi = \frac{R^2 \cdot U_\infty \cdot \cos \theta}{r} - U_\infty \cdot r \cdot \cos \theta = R \cdot U_\infty \left( \frac{R}{r} - \frac{r}{R} \right) \cdot \cos \theta \]  \hspace{1cm} (3.91)

The hydrodynamics of a circular cylinder is obviously \textit{very significant} for offshore engineering! The relevant flow pattern is shown in figure 3.12.

It is worthwhile to 'play' with equation 3.84 a bit more. This leads to the following discoveries:

1. Far from the cylinder, when \( x^2 + y^2 \) is large, the stream lines becomes \( \Psi_\infty = U_\infty \cdot y \).
3.5. SUPERPOSITION OF BASIC ELEMENTS

One also has to check that there is no radial velocity on the cylinder surface. This implies that the following relationship is satisfied - see 3.52-b:

\[ v_r = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \bigg|_{r=R} = 0 \]  

(3.92)

This gives no problems, either.

One can also check that the potential function fulfills the Laplace equation \((\nabla^2 \Phi = 0)\).

Further details of the flow around a cylinder will be discussed later in this chapter. One additional application of superposition is discussed here first.

3.5.8 Pipeline Near The Sea Bed

Figure 3.12 shows the flow around a single cylinder. Superposition can be used to simulate (approximately) the flow around a cylinder near the (flat) sea bed. The reason why this is approximate will become clear later in this section. Since the sea bed is flat and impervious, it must be a straight streamline. A straight streamline parallel to the \(x\) axis at some given distance from the cylinder axis, \(y_0\) from the cylinder is needed. This can be achieved (approximately) by building up a symmetrical pattern involving two cylinders, one at a distance \(y_0\) above the \(x\) axis and one at a distance \(y_0\) below that axis as shown in figure 3.13. The lower half of the flow pattern is just 'thrown away' by ignoring it. The sea bed will be the \(x\) axis with the \(y\) axis positive upward passing through the...
cylinder center. This special form of superposition is often referred to as reflection (about the sea bed, in this case). Reflection is commonly used for other applications with the sea surface or even a vertical plane of symmetry as the reflection plane.

\[
\Psi = \mu \cdot \frac{y - y_0}{x^2 + (y - y_0)^2} + \mu \cdot \frac{y + y_0}{x^2 + (y + y_0)^2} - U_\infty \cdot y \quad \text{(3.93)}
\]

By letting \( y \) equal zero (at the sea bed) in this equation, one can check that \( \Psi \) remains constant and equal to zero in order to prove that the sea bed is indeed a streamline; this is left to the reader.

A more interesting (and more complicated) check is to see if the surface of the pipeline \( R^2 = x^2 + (y - y_0)^2 \) is still a streamline and thus is (still) impermeable. A quick way to disprove this can be to check the value of \( \Psi \) at a few convenient points on the circle representing the pipeline. If the values of \( \Psi \) computed at these points on the circle are not identical, then the circle can no longer be a stream line. It can be convenient to select locations such as the top, bottom, front and back of the pipe for this check as is done in the table below.

<table>
<thead>
<tr>
<th>Location</th>
<th>( x )</th>
<th>( y )</th>
<th>( \Psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>0</td>
<td>( y_0 + R )</td>
<td>( \mu \cdot \left( \frac{1}{R} + \frac{1}{2y_0 + R} \right) - U_\infty \cdot (y_0 + R) )</td>
</tr>
<tr>
<td>Front</td>
<td>( +R )</td>
<td>( y_0 )</td>
<td>( \mu \cdot \frac{2y_0}{R^2 + 4y_0^2} - U_\infty \cdot y_0 )</td>
</tr>
<tr>
<td>Bottom</td>
<td>0</td>
<td>( y_0 - R )</td>
<td>( \mu \cdot \left( \frac{1}{R} + \frac{1}{2y_0 - R} \right) - U_\infty \cdot (y_0 - R) )</td>
</tr>
<tr>
<td>Back</td>
<td>( -R )</td>
<td>( y_0 )</td>
<td>( \mu \cdot \frac{2y_0}{R^2 + 4y_0^2} - U_\infty \cdot y_0 )</td>
</tr>
</tbody>
</table>

One can quickly see from this table that the pipeline’s circular surface is no longer a streamline. Why should this be so? The answer lies in the fact that the influence from each potential flow element extends (theoretically) to infinity. Since it has been shown above that a single doublet pair in a uniform flow does yield a perfect circular section, then it is only to be expected that this circular streamline will be disturbed a bit by the influence of the second doublet introduced in order to make a symmetric pattern with respect to the sea bed. This is why the word 'approximate' was used so rigorously at the beginning of this discussion.
3.6 Single Cylinder in a Uniform Flow

A cylinder is such an important element in offshore structures that its hydrodynamics will be studied in more detail in this section.

3.6.1 Flow

Velocity along Cylinder Wall

It is easiest to determine the tangential velocity along the wall of a cylinder using polar coordinates. Using equation 3.52, one finds that:

\[ v_\theta = - \left[ \frac{\partial \Psi}{\partial r} \right]_{r=R} = - \frac{\partial}{\partial r} \left\{ \frac{\mu \sin \theta}{r} + U_\infty r \sin \theta \right\} \]

which, with \( \mu = U_\infty R^2 \), yields:

\[ v_\theta = -2U_\infty \sin \theta \]

Observe that:

\[ v_\theta = 0 \text{ for: } \theta = n\pi \]

The tangential velocity where the cylinder wall crosses the \( x \) axis is zero. Also:

\[ v_\theta = 2U_\infty \text{ for: } \theta = (n + \frac{1}{2})\pi \]

The tangential velocity at the 'sides' of the cylinder (where the wall crosses the \( y \) axis) is exactly twice the undisturbed ambient flow velocity. Note also that this ratio \( \frac{v_\theta}{U_\infty} \) is independent of the cylinder radius. In theory, the velocity adjacent to a human hair would be just as big as that adjacent to a large circular pile; significant development of micro-scale velocity measurement techniques will be needed before this can be proven, however!

In the two above equations:

- \( U_\infty = \) Undisturbed ambient flow velocity magnitude \((m/s)\)
- \( v_\theta = \) Velocity component along the cylinder wall \((m/s)\)
- \( n = \) Any integer value

Velocity Profile Adjacent to Cylinder

It can be instructive to examine the velocity distribution along the \( y \) axis \((x = 0)\):

\[ u = \left[ \frac{\partial \Psi}{\partial y} \right]_{x=0} = U_\infty \cdot \left[ \frac{R^2}{y^2} + 1 \right] \]

\( u \) decreases to \( U_\infty \) inversely with \([\frac{y}{R}]^2\). It is equal to \( 2U_\infty \) for \( y = R \) just as was found above; this value is worth remembering by the way!
Influence of Circulation

If a circulation is added to the flow, then $v_\theta$ along the cylinder wall comes directly from the superposition of equations 3.95 and 3.69:

$$v_\theta = 2U_\infty \sin \theta + \frac{\Gamma}{2\pi R}$$  \hspace{1cm} (3.99)\]

again at the cylinder surface. Notice that $v_\theta$ is no longer symmetric with respect to the $x$ axis, as shown in figure 3.14. In this case, a clockwise circulation has been added to a flow from left to right. This, too, will be important for the hydrodynamic force discussion in the next main section of this chapter. A discussion of how this circulation is generated comes up later in this chapter.

![Figure 3.14: Cylinder in Uniform Flow with added Circulation](image)

3.6.2 Pressures

Now that the potential flow around the cylinder is known, it is appropriate to discuss pressures and the forces which result from them. This treatment starts, again, with a single isolated cylinder in an infinite flow; there is no circulation.

Stagnation Points

It has already been pointed out via equation 3.96 that the tangential velocity component at the two locations where the cylinder wall and the $x$ axis cross is identically equal to zero. Since the cylinder wall is impervious, the radial velocity component, $v_r$, at this location is also zero. There is therefore no velocity at these two stagnation points. These points were labeled in figure 3.12.

Stagnation pressures are generally defined in terms of the ambient flow velocity so that:

$$p_s = \frac{1}{2}\rho U_\infty^2$$  \hspace{1cm} (3.100)
3.6. SINGLE CYLINDER IN A UNIFORM FLOW

Pressure Distribution on Cylinder Wall

The next step is to predict the complete pressure distribution along the entire circumference of the cylinder. This is done using the Bernoulli equation 3.44 and the known pressure and velocity at the stagnation point:

\[ \frac{1}{2} \rho U_\infty^2 + 0 = p + \frac{1}{2} \rho v_\theta^2 = \text{const.} \] (3.101)

The elevation is left out of this balance because it is constant here. Equation 3.95 can be substituted for \( v_\theta \) here and equation 3.101 can be solved for the pressure, \( p \), along the cylinder wall. This yields:

\[ p = \frac{1}{2} \rho U_\infty^2 \cdot \left[ 1 - 4 \sin^2 \theta \right] \] (3.102)

This (dimensionless) pressure distribution (the term in brackets above) is shown in figure 3.15 in polar coordinates. It is also plotted in a rectangular coordinate form in chapter 4, by the way.

![Figure 3.15: Pressure Distribution on a Cylinder in a Potential Flow](image)

3.6.3 Resulting Forces

The next step is to determine the forces from the pressure distribution.

Force Computation Principle

The procedure used to compute the \( x \) and \( y \) components of the resulting force on the cylinder is illustrated for the \( x \) direction using figure 3.16.

This figure shows a 'slice' which extends from front to back across the cylinder; it isolates an arc length \( ds = R d\theta \) of the cylinder wall. The pressure on each end of this slice is directed normal to the surface (radially) so that its \( x \)-component is then \( dF_x = p \cdot ds \cos \theta \).

Now, one only has to integrate this over the angle \( \theta \) to get the total force \( F_x \).
D’Alembert’s Paradox

The French mathematician Jean LeRond D’Alembert who died in 1783, is credited with having carried out the above computation and with discovering that \( F_x = 0 \) and - from an analogous computation in the \( y \) direction - \( F_y = 0 \) for a circular cylinder placed in a constant, uniform, potential flow! This is D’Alembert’s paradox - a trap into which most every beginner falls. It could have been avoided by observing - beforehand! - that the pressure distribution in figure 3.15 is completely symmetric about both the \( x \) and \( y \) axes so that any forces must cancel out in each of these directions. The conclusion should be very clear: There are no hydrodynamic forces on a circular cylinder in a constant uniform, potential flow.

Circulation and Lift

If a circulation is added to the flow used in these computations, a force will be generated. This comes about because the circulation will increase the tangential flow velocity on one side of the cylinder - say the \(+y\) side - while it decreases the flow velocity on the opposite - in this case \(-y\) side; see figure 3.14. This is often referred to as the Magnus effect. This extra flow will obviously unbalance the symmetry of the flow with respect to the \( x \)-axis; one could expect the pressure to be disturbed, too, so that a resultant force in the \( y \) direction will result.

The tangential velocity will now be as given in equation 3.99. By letting

\[
C = -\frac{\Gamma}{4\pi RU_\infty} = constant
\]  

(3.103)

then 3.99 becomes:

\[
v_\theta = 2U_\infty \cdot (\sin \theta - C)
\]  

(3.104)

and the equation for the pressure distribution, 3.102, now becomes:

\[
p = \frac{1}{2} \rho U_\infty^2 \left\{ 1 - 4(\sin \theta - C)^2 \right\}
\]  

(3.105)
The drag force can be evaluated just as was done above; this still yields a zero resultant, independent of the value of $C$.

The resulting force crosswise to the flow can be found by integrating the $y$-component of the pressure force as follows:

$$F_y = R \int_0^{2\pi} p \sin \theta \, d\theta$$  \hspace{1cm} (3.106)

After a bit of substitution and algebra one should find that:

$$F_y = 4\pi R \rho U_\infty^2 C$$  \hspace{1cm} (3.107)

This resultant force is called lift; it is directed perpendicular to the direction of the undisturbed approaching flow. Notice that the lift force is only present when $C \neq 0$.

Lift forces are responsible for keeping aircraft airborne. The necessary tangential velocity differential is achieved by making the flow path over the top of a wing or helicopter rotor longer than the flow path on the underside so that the flow over the top must travel faster to pass the object in the same time interval. This same phenomenon is important with ship’s propellers, too; see chapter 4.

Lift is largely responsible for driving a sailboat forward when it is sailing ’close-hauled’. The necessary low pressure ’behind’ the main sail is now created by the jet caused by the jib or foresail. In tennis or with a baseball pitch, a spinning ball will curve one way or another. Now the ambient air remains stationary while the ball speeds through it. The necessary circulation is created by friction between the spinning ball and the air. Since friction is involved in this case, discussion will have to be delayed until the next chapter.
Chapter 4

CONSTANT REAL FLOW PHENOMENA

4.1 Introduction

Now that the basics of constant idealized potential flows have been discussed in the previous chapter, the next step is to consider more realistic - but still constant - flows. The facts that there is no drag force in a potential flow and that a circulation is required to generate a lift force is true only for a potential (non-viscous) flow. The results from this chapter with real flows which include viscous effects will be more realistic.

The initial discussion of some basic concepts motivates attention for two common dimensionless numbers, $Rn$ and $Fn$, which are handled more generally in Appendix B. These are used to characterize constant flows around cylinders and ships - and the resultant forces - discussed in the latter part of the chapter. Propulsion systems are handled toward the end of the chapter.

4.2 Basic Viscous Flow Concepts

This section discusses a few basic principles of constant viscous flows. These supplement the discussion on potential flows given in the first parts of chapter 3.

4.2.1 Boundary Layer and Viscosity

In 1883, Osborne Reynolds, an English investigator, observed the flow of water through a glass pipe which drained a container water as shown somewhat schematically in figure 4.1. He first filled the tank with the drain closed so that the water could come to rest.

After opening the drain valve, he found that the flow and streamlines - visualized by a stream of ink:

- The streamlines were straight and parallel in the first segment of the pipe just downstream from its entrance.
- The flow velocity near the pipe wall was less than that near the axis of the pipe.

The streamlines eventually became irregular as the flow progressed further along the pipe. This first happened nearer the mouth of the pipe for flow near its walls and further downstream for flow along the pipe center line.

Further, he observed that if he kept the valve nearly closed (so that all velocities were low) the above development process (as a function of distance in this case) was less abrupt. The flow progressed further before the entire flow became irregular. If the flow velocity was higher, then the whole process took place in a shorter length of pipe.

By repeating the experiments with other fluids and with other pipe diameters, Reynolds discovered that similar phenomena were observed when the following dimensionless parameter had a constant value:

\[ R_n = \frac{V \cdot D}{\nu} \] (4.1)

in which:

- \( R_n \) = Reynolds number (-)
- \( V \) = flow velocity (m/s)
- \( D \) = pipe diameter (m)
- \( \nu \) = kinematic viscosity of the fluid (m²/s)

The kinematic viscosity is a fluid property usually quite dependent upon temperature. For water at room temperature, its value is about \( 1 \cdot 10^{-6} \) m²/s. This value can be reduced by a factor of two or so simply by heating the water - see appendix A.

The Reynolds number is usually interpreted physically as the ratio of Inertia Forces to Viscous Forces. Viscous forces are most important, therefore, when the Reynolds number is small.

### 4.2.2 Turbulence

The breakup of the nice neat streamlines far enough downstream in Reynolds’ experimental pipeline was caused by what is called turbulence. Turbulence is a result of small eddies or vortices which form in most any viscous flow. Turbulence manifests itself in a flow measurement by causing time-dependent fluctuations in the velocity measured at one point, but in such a way that the average velocity remains unchanged.
Turbulent eddies can have many scales and occur in all sorts of flows. In the atmosphere, a sensitive wind speed meter (anemometer) can register rather quick turbulent fluctuations in the flow. On a larger scale, a wind gust can be seen as a larger scale of turbulence. On a global scale, even a whole hurricane can be considered to be turbulence, relative to the even larger global atmospheric circulation.

### 4.2.3 Newton’s Friction Force Description

Reynolds did not study fluid forces in his experiments. Newton had already done this for fluids as well as for solids; although he is most well-known for his work in solid mechanics. Newton’s postulated a fluid friction law based upon the following model. Consider two parallel plates, each of unit area, and separated by a fluid filling a width $y$ between the plates as shown in figure 4.2.

![Figure 4.2: Newton’s Friction Postulation](image)

If the upper plate is moved with a velocity, $V$, with respect to the other, then Newton postulated that the velocity profile between the plates was linear, and that the force needed to sustain the movement was proportional to:

$$\tau = \eta \cdot \frac{V}{y} \quad \text{or} \quad \tau = \eta \cdot \frac{dV}{dy}$$

(4.2)

in which:

- $\eta$ = Dynamic viscosity of the fluid ($\frac{kg}{m \cdot s}$)
- $\tau$ = Shear force per unit area or shear stress ($N/m^2$)
- $\frac{dV}{dy}$ = Velocity gradient (1/s)

Equation 4.2 works well for viscous fluids and for low velocities and thus Reynolds numbers smaller than about 2000. Other formulations for the fluid friction force under more realistic situations are given in chapter 14.

There is a straightforward relationship between the dynamic viscosity, $\eta$, and kinematic viscosity, $\nu$:

$$\nu = \frac{\eta}{\rho}$$

(4.3)

in which $\rho$ ($kg/m^3$) is the mass density of the fluid.
4.3 Dimensionless Ratios and Scaling Laws

One dimensionless ratio - the Reynolds number - has already been introduced above. This section discusses this and one additional common dimensionless number: the Froude number. These are both commonly found in offshore hydromechanics.

Dimensionless ratios are used to characterize families of different conditions in an efficient way. Reynolds used a single number above to characterize flows of different fluids through different diameter pipes and at different velocities. His Reynolds number conveniently combined the three independent variables to form a single one. As long as a consistent set of units is used - it makes no difference if the velocity has been measured in meters per second, feet per minute, or even leagues per fortnight!

For example, by using the Reynolds number to characterize the flow in a pipeline, it is possible to calibrate a venturi or orifice meter using a flow of water even though the meter may later be used to measure a flow of something more exotic like supercritical steam or even molten sodium.

A second use of dimensionless numbers is to correlate (or convert) measurements made on small physical models to equivalent values for a full-sized or prototype situation. This allows experiments to be carried out on a series of relatively inexpensive physical models rather than having to build as many full-sized ships (or other objects) instead.

Dimensionless numbers will be used for both purposes later in this chapter. The remainder of this section is concerned primarily with physical models, however.

4.3.1 Physical Model Relationships

Physical model experiments require some form of similarity between the prototype and the model:

- **Geometric similarity**: The model must have physical dimensions which are uniformly proportional to those of the prototype; it must have the same shape.

- **Kinematic similarity**: Velocities in the model must be proportional to those in the prototype.

- **Dynamic similarity**: Forces and accelerations in the model must be proportional to those in the prototype.

These three similarities require that all location vectors, velocity vectors and force vectors in the coincident coordinates of the scaled model and the prototype have the same direction (argument) and that the magnitude of these vectors (modulus) must relate to each other in a constant proportion.

If \( \alpha \) - a number larger than 1 - is used to denote the ratio between the prototype (subscript \( p \)) quantity and the model (subscript \( m \)) quantity, then:
### 4.3 DIMENSIONLESS RATIOS AND SCALING LAWS

<table>
<thead>
<tr>
<th>Item</th>
<th>Scale Factor</th>
<th>Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>$\alpha_L$</td>
<td>$L_p = \alpha_L \cdot L_m$</td>
</tr>
<tr>
<td>Velocity</td>
<td>$\alpha_V$</td>
<td>$V_p = \alpha_V \cdot V_m$</td>
</tr>
<tr>
<td>Acceleration of gravity</td>
<td>$\alpha_g$</td>
<td>$g_p = \alpha_g \cdot g_m$</td>
</tr>
<tr>
<td>Density of fluid</td>
<td>$\alpha_\rho$</td>
<td>$\rho_p = \alpha_\rho \cdot \rho_m$</td>
</tr>
<tr>
<td>Fluid viscosity</td>
<td>$\alpha_\eta$</td>
<td>$\eta_p = \alpha_\eta \cdot \eta_m$</td>
</tr>
</tbody>
</table>

With these, the scale factors for the areas $S$, the volumes $\nabla$, the masses $M$ and the mass moments of inertia $I$, are respectively:

$$
\alpha_S = \alpha_L^2 \quad \alpha_\nabla = \alpha_L^3 \quad \alpha_I = \alpha_\rho \cdot \alpha_L^5
$$

$$
\alpha_M = \alpha_\rho \cdot \alpha_\nabla = \alpha_\rho \cdot \alpha_L^3
$$

(4.4)

The velocity of a body or a water particle is defined as a displacement per unit of time, so the scale factor for the time becomes:

$$
\alpha_T = \frac{\alpha_L}{\alpha_V}
$$

(4.5)

The acceleration of a body or a water particle is defined as an increase of the velocity per unit of time, so the scale factor for the acceleration becomes:

$$
\alpha_A = \frac{\alpha_V}{\alpha_T} = \frac{\alpha_L^2}{\alpha_L}
$$

(4.6)

According to Newton’s law, the inertia forces are defined as a product of mass and acceleration, so the scale factor for the inertia forces (and the resulting pressure forces) works out to be:

$$
\alpha_F = \alpha_M \cdot \alpha_A = \left(\alpha_\rho \cdot \alpha_L^3\right) \cdot \left(\frac{\alpha_L^2}{\alpha_L}\right)
$$

$$
= \alpha_\rho \cdot \alpha_V^2 \cdot \alpha_L^2
$$

(4.7)

Then, the relation between the forces $F_p$ on the prototype and the forces $F_m$ on the model is:

$$
F_p = \alpha_F \cdot F_m
$$

$$
= \alpha_\rho \cdot \alpha_V^2 \cdot \alpha_L^2 \cdot F_m
$$

(4.8)

or:

$$
\alpha_F = \frac{F_p}{F_m} = \frac{\rho_p \cdot V_p^2 \cdot L_p^2}{\rho_m \cdot V_m^2 \cdot L_m^2}
$$

(4.9)

From this, it is obvious that one can write for these forces:

$$
F_p = C \cdot \frac{1}{2} \rho_p V_p^2 \cdot L_p^2 \quad \text{and} \quad F_m = C \cdot \frac{1}{2} \rho_m V_m^2 \cdot L_m^2
$$

(4.10)
in which the constant coefficient, $C$, does not depend on the scale of the model nor on the stagnation pressure term $\frac{1}{2} \rho V^2$.

Viscous forces can be expressed using Newton’s friction model as being proportional to:

$$F_v \propto \frac{V}{\eta L} L^2$$  \hspace{1cm} (4.11)

while the inertia forces (from above) are proportional to:

$$F_i \propto \rho L^3 \frac{V^2}{L} = \rho L^2 V^2$$  \hspace{1cm} (4.12)

The ratio of these two forces is then - after cancelling out some terms:

$$\frac{F_i}{F_v} = \frac{\rho}{\eta} V \cdot L = \frac{V \cdot L}{\nu} = Rn$$  \hspace{1cm} (4.13)

The Reynolds number is thus a measure of the ratio of these forces. Viscous forces are predominant when the Reynolds number is small.

Gravity forces are simply proportional to the material density, the acceleration of gravity and the volume:

$$F_g \propto \rho g L^3$$  \hspace{1cm} (4.14)

The above information can be used to help design physical models. One can deduce that various forces are represented to different scales. It is therefore impossible to represent all model forces with the same relative importance as in the prototype. A choice is therefore often made, instead, to maintain the ratio between the two most important forces in the prototype when a physical model is built. This can result in several scaling laws as outlined in appendix B. Two of the more important forms are discussed here, however.

### 4.3.2 Reynolds Scaling

Reynolds scaling is used when inertia and viscous forces are of predominant importance in the flow. This is the case for pipe flow (under pressure) and for wake formation behind a body in a flow. Reynolds scaling requires that the Reynolds number in the model be identical to that in the prototype. Using the basic relations above:

$$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p}$$  \hspace{1cm} (4.15)

or, expressed another way:

$$\frac{\alpha_V \cdot \alpha_L}{\alpha_\nu} = 1.$$  \hspace{1cm} (4.16)

Since $\alpha_V = \alpha_L/\alpha_T$, then this becomes:

$$\alpha_T = \frac{\alpha_T^2}{\alpha_\nu}$$  \hspace{1cm} (4.17)

Since the fluids in the model and in the ocean are about the same, $\alpha_\nu$ (and even $\alpha_\rho$ for that matter) is very close to 1.0, so that:

$$\alpha_T \approx \alpha_L^2$$  \hspace{1cm} (4.18)
This last result, that the time scale is the square of the length scale, would mean that time would pass quite rapidly in the model. Also, one can make further substitutions to discover that $\alpha_T$ is equal to unity; forces in the model would be just as big as in the prototype - again very impractical!

### 4.3.3 Froude Scaling

Gravity forces become important when a free surface of a liquid is involved. This will be true, whenever a water surface or waves are present. Since inertia and pressure forces are nearly universally important, this makes it appropriate to keep the ratio of \( \frac{\text{inertia or pressure forces}}{\text{gravity forces}} \) the same in the model as in the prototype. Scaling based upon the square root of this ratio is called Froude scaling, after Robert Edmund Froude (as distinct from his father William Froude from the model resistance extrapolation to full scale, treated in a following section) who has first used it.

Working this out from the basic information above and in a way very analogous to that used for Reynolds scaling yields:

\[
F_n = \sqrt{\frac{\alpha\cdot\alpha_T^2\cdot \alpha_p^2}{\alpha\cdot\alpha_p^2\cdot\alpha_g}} = \frac{\alpha_V}{\sqrt{\alpha_g\cdot\alpha_L}} = \frac{V}{\sqrt{gL}} \tag{4.19}
\]

Since it is especially difficult (or at least very costly) to change the acceleration of gravity for a model involving free liquid surfaces, one may safely set $\alpha_g = 1$. Continuing as was done above, one finds that:

\[
\alpha_T = \sqrt{\alpha_L} \tag{4.20}
\]

This is a lot more convenient to handle than Reynolds scaling!

### 4.3.4 Numerical Example

As an example, suppose a ship with a length $L_s = L_{pp} = 100$ meter, which sails with a forward ship speed $V$ of 20 knots in still seawater with a temperature of 15°C. Resistance and propulsion tests will be carried out in a towing tank with a 1:40 scale physical model ($\alpha_L = 40$). The temperature of the fresh water in the tank is 20°C.

The density and the kinematic viscosity of fresh water, salt water and dry air as a function of the temperature are listed in appendix A. Values needed here are for sea water: $\rho = 1025.9$ kg/m$^3$ and $\nu = 1.19 \cdot 10^{-6}$ m$^2$/s. The relevant values for fresh water are now: $\rho = 998.1$ kg/m$^3$ and $\nu = 1.05 \cdot 10^{-6}$ m$^2$/s.

The length of the ship model is:

\[
L_m = \frac{L_s}{\alpha_L} = \frac{100}{40} = 2.50 \text{ m} \tag{4.21}
\]

The speed $V_s$ of the ship is:

\[
V_s = 0.5144 \cdot V = 0.5144 \cdot 20 = 10.29 \text{ m/s} \tag{4.22}
\]
CHAPTER 4. CONSTANT REAL FLOW PHENOMENA

For practical reasons, the speed of the model will be obtained using Froude scaling:

\[ V_m = \frac{V_s}{\sqrt{\alpha_L}} = \frac{10.29}{\sqrt{40}} = 1.63 \text{ m/s} \quad (4.23) \]

As a consequence, the Froude numbers for the ship and model are obviously identical and equal to 0.329.

A consequence of this scaling is that the Reynolds numbers will differ:

\[ Rn_s = \frac{V_s \cdot L_s \cdot \nu_{salt}}{1.19 \cdot 10^{-6}} = 865 \cdot 10^6 \]
\[ Rn_m = \frac{V_m \cdot L_m \cdot \nu_{fresh}}{1.05 \cdot 10^{-6}} = 3.88 \cdot 10^6 \quad (4.24) \]

so that:

\[ \alpha_{Rn} = 223 \quad (4.25) \]

In order to obtain equal Reynolds numbers, the "model water" needs a kinematic viscosity which is 1/223 times its actual value; this liquid is not available!

Reynolds versus Froude scaling will be picked up again later in this chapter. First, however, the next sections of this chapter will discuss the flow around and hydrodynamic forces on a slender cylinder in a constant current.

### 4.4 Cylinder Flow Regimes

This figure below summarizes the various forms of flow around a cylinder. A more complete discussion can be found in [Schlichting, 1951]. The Reynolds number (listed on the left, below the flow pattern name) is used in figure 4.3 to characterize each type of flow. The significance of the \( Cd \) values listed there will become clear in a later section of this chapter, by the way. The right hand column in figure 4.3 gives a short comparative description of the flow pattern.

### 4.5 Drag and Lift

Now that the flow patterns near a circular cylinder have been determined, attention can be shifted to the forces that result from the flow.

#### 4.5.1 Drag Force and Drag Coefficient

The pressure variation near a cylinder wall was given in chapter 3. Applying this now in a slightly different way to reveal the pressure difference, relative to an undisturbed flow pressure, yields:

\[ \Delta p = p - p_0 = \frac{1}{2} \cdot \rho \cdot U_\infty^2 \cdot (1 - 4 \cdot \sin^2 \theta) \quad (4.26) \]

in which:
<table>
<thead>
<tr>
<th>Flow and Parameters</th>
<th>Flow Pattern</th>
<th>Flow Description</th>
</tr>
</thead>
</table>
| **1. Viscous**<br>
$Rn < 1$<br>$C_D > 12$ | ![Flow Pattern Image](image1) | Flow is symmetric about both axes. Friction dominates. |
| **2. Hele-Shaw**<br>$1 < Rn < 5$<br>$12 > C_D > 4.5$ | | Slight separation downstream. Symmetry about flow axis only. Inertia becomes important. |
| **3. Symmetric Vortices**<br>$5 < Rn < 65$<br>$4.5 > C_D > 1.6$ | ![Flow Pattern Image](image2) | First separation and vortices downstream. Wake unstable (turbulent) farther downstream. |
| **4. Precritical**<br>$65 < Rn < 5000$<br>$1.6 > C_D > 1.2$ | | Vortices become unstable and alternately separate forming a Von Karman vortex street. |
| **5. Subcritical**<br>$5K < Rn < 200K$<br>$C_D$ about 1.2 | | Turbulent mixing between vortices downstream. Laminar boundary layer on front of cylinder. |
| **6. Critical**<br>$200K < Rn < 500K$<br>$1.2 > C_D > 0.3$ | | Turbulent wake behind cylinder. Separation and re-attachment of boundary layer upstream of wake. |
| **7. Supercritical**<br>$500K < Rn < 4M$<br>$0.3 < C_D < 0.7$ | | Separation 'bubble' gone. Cylinder boundary layer begins to be turbulent. Wake becomes wider. |
| **8. Postcritical**<br>$Rn > 4M$<br>$C_D$ about 0.7 | | Turbulence reaches the boundary layer on the upstream side. |

Figure 4.3: Flow Regimes and Descriptions
\( \Delta p = \) pressure difference (\( \frac{\Delta \rho}{m} = N/m \))
\( p = \) local pressure at the cylinder wall (N/m)
\( p_0 = \) ambient pressure in the undisturbed flow (N/m)
\( U_\infty = \) undisturbed flow velocity (m/s)
\( \rho = \) mass density of the fluid (kg/m\(^3\))
\( \theta = \) angle measured from approaching flow direction (rad)

Since the term \( \frac{1}{2} \cdot \rho \cdot U_\infty^2 \) also has units of pressure (it is in fact the stagnation pressure), it can be convenient to make \( \Delta p \) dimensionless by dividing it by this value. This defines the dimensionless pressure coefficient, \( C_p \). In potential flow theory this coefficient has a value of:

\[
C_p = \frac{\Delta p}{\frac{1}{2} \cdot \rho \cdot U_\infty^2} = (1 - 4 \cdot \sin^2 \theta)
\] (4.27)

Experimental and theoretical \( C_p \) values are compared in figure 4.4 for smooth cylinders. (The theoretical value of \( C_p \) was plotted in polar coordinates in chapter 3, by the way.)

![Figure 4.4: Comparison of Theoretical and Experimental Pressure Coefficients](image)

The experimental results compare well to the theory only for small values of \( \theta \) (on the upstream side). \( C_p \) remains approximately constant for \( \theta \) values greater than about 80 to 120 degrees depending on the Reynolds number. The disagreement between experiments and theory is an indication that the flow has at least been disturbed and generally has separated. The flow disturbance results in a drag force - by definition parallel to the flow.
direction - on the cylinder. This force (on a unit length of cylinder) can be written as a function of the (undisturbed) stagnation pressure:

\[ F_D = \frac{1}{2} \rho U^2 C_D D \]  

(4.28)

in which:

- \( F_D \) = drag force per unit length of cylinder (N/m)
- \( \rho \) = mass density of the fluid (kg/m³)
- \( C_D \) = dimensionless coefficient (-)
- \( D \) = cylinder diameter (m)
- \( U \) = undisturbed flow velocity (m/s)

The drag force is the product of the stagnation pressure, \( \frac{1}{2} \rho U^2 \), an area, \( D \) times a unit length, and a dimensionless coefficient, \( C_D \).

This equation thus defines the drag coefficient as:

\[ C_D = \frac{F_D}{\frac{1}{2} \rho U^2 \cdot D} \]  

(4.29)

The drag coefficient depends on \( Rn \), the cylinder’s roughness and the turbulence of the incident flow. The resulting drag coefficients for a fixed smooth cylinder in a constant velocity flow have already been included in figure 4.3.

**Fall Velocity**

One of the simplest applications of the drag force concept is for the determination of the fall velocity of an object. This is the maximum speed with which an object will fall through the medium which surrounds it. A parachute jumper soon reaches his or her fall velocity (of about 7 m/s) after jumping out of an aircraft, for example. Attention in offshore engineering focuses instead on the speed at which sand grains or gravel or even ropes and chains fall through sea water.

The fall velocity in any of these cases is determined by an equilibrium between the submerged weight (see chapter 2) of the object on the one hand and the drag force from its environment on the other. In equation form:

\[ W_{sub} = C_D D \cdot \frac{1}{2} \rho V_f^2 \]  

(4.30)

in which \( W_{sub} \) is the submerged weight of the object (N) and \( V_f \) is the fall velocity of the object (m/s).

If one is considering an isolated particle - such as a stone, then the drag coefficient, \( C_D \), used in equation 4.30 will be for the three-dimensional object. If, on the other hand, a long object such as a rope (fall crosswise) is being considered, then \( W_{sub} \) and \( C_D \) relate to the two dimensional object.

\( C_D \) in equation 4.30 is dependent upon the Reynolds number which depends in turn upon the velocity, \( V_f \). One must iterate (a simple trial-and-error approach is sophisticated enough) to find the proper combination of these two unknowns. This is illustrated here for a steel wire rope such as is used as the main tow rope on the *Smit Rotterdam*, one of
the largest tugboats in the world. The basic data is that the segment of wire rope is 3 inches (76 mm) in diameter and has a submerged weight of 196 N/m. Because the surface of the rope is rough, (it is made from twisted strands of wire) it will have a drag coefficient which is about 30% higher than that listed above for a smooth cylinder. The iteration is indicated in the following table.

\[
\begin{array}{cccc}
 V_f \text{ guess} & Rn & C_D & V_f \text{ calc} \\
 (\text{m/s}) & (-) & (-) & (\text{m/s}) \\
 5.0 & 375 000 & 1.2 & 2.05 \\
 2.0 & 150 000 & 1.6 & 1.78 \\
 1.78 & 131 250 & 1.6 & 1.78 \\
\end{array}
\]

The computation converges quite easily. The numerical value may surprise some; even a seemingly heavy steel wire rope will not fall all that fast in water!

The order of magnitude of the fall velocities for other materials is summarized in the table below.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>D (mm)</th>
<th>$W_{\text{sub}}$ (N)</th>
<th>$Rn$ (-)</th>
<th>$V_f$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Object</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sand grain</td>
<td>0.2</td>
<td>$0.07 \cdot 10^{-6}$</td>
<td>$\approx 2$</td>
<td>0.02</td>
</tr>
<tr>
<td>gravel</td>
<td>20</td>
<td>0.07</td>
<td>20 000</td>
<td>1.00</td>
</tr>
<tr>
<td>stone</td>
<td>100</td>
<td>8.38</td>
<td>235 000</td>
<td>2.35</td>
</tr>
<tr>
<td>1 m wire rope</td>
<td>76</td>
<td>196</td>
<td>131 250</td>
<td>1.78</td>
</tr>
<tr>
<td>1 m chain</td>
<td>76</td>
<td>1079</td>
<td>396000</td>
<td>3.00</td>
</tr>
</tbody>
</table>

This data for sand, gravel and stones will be handy later when sea bed morphology is discussed in chapter 14. The diameter used for chain is based upon its mass per unit length. Its drag coefficient is about 50% higher than that of a smooth cylinder. Of course some objects can fall much faster than the values given here. A 400 ton pile was dropped by accident in the Gulf of Mexico when it was hanging in a vertical position. A crude underwater measurement, made before it would have reached its terminal fall velocity, indicated a speed of about 11 m/s. When it hit the sea bed - luckily next to the platform it was originally intended for - its impact caused it to penetrate 75 meters into the sea bed.

### 4.5.2 Lift Force and Strouhal Number

A **lift force** is defined as a force component acting perpendicular to the undisturbed flow velocity. It is therefore also perpendicular to the drag force component. As explained
4.5. DRAG AND LIFT

in chapter 3, there is no lift force when the cylinder is placed in a completely symmetric potential flow. Since the flow pattern remains symmetric in a real flow for Reynolds numbers below about 65, there will be no lift force under these conditions either. However, from the time that vortices are (alternately) shed behind the cylinder, a cyclic pressure variation will occur in the wake. At the location where the vortex is closest to the cylinder the local velocities in the wake will be highest and the local pressures will be lowest. This leads to a resulting force directed toward the vortex; its component in the flow direction is the drag force discussed above, its component perpendicular to the flow direction is the lift. See figure 4.5.

![Figure 4.5: Lift, Drag and Resultant Forces](image)

Because vortices are shed alternately, this lift force will alternate in direction as well. Its magnitude will vary more or less sinusoidally with a frequency corresponding to the vortex shedding frequency, \( f_v \), so that:

\[
F_\ell = \frac{1}{2} \rho U^2 \cdot D \cdot C_L \cdot \sin\left(2 \pi f_v t + \varepsilon_{Fl}\right)
\]

in which:

- \( F_\ell \) = lift force per unit cylinder length (N/m)
- \( f_v \) = vortex shedding frequency (Hz)
- \( D \) = cylinder diameter (m)
- \( U \) = undisturbed flow velocity (m/s)
- \( C_L \) = dimensionless lift coefficient (-)
- \( t \) = time (s)
- \( \varepsilon_{Fl} \) = phase shift (rad)

Given the vortex shedding frequency, one can define the dimensionless **Strouhal number** as:

\[
St = \frac{f_v \cdot D}{U}
\]

where \( St \) depends on the Reynolds number, \( Rn \), as shown in figure 4.6 taken from reference [Lienhard, 1966] for circular cylinders. It is valid for a circular cylinder with its axis perpendicular to the flow.

Notice that for \( 10^2 < Rn < 2 \cdot 10^5 \) the Strouhal number, \( St \), is approximately 0.2. For \( Rn > 2 \cdot 10^5 \) the Strouhal number, \( St \), increases a little bit; the vortices and thus the lift force fluctuations are no longer regular and its frequency becomes harder to define.
Figure 4.7 schematically shows the vortex street behind a cylinder at four consecutive time instants during a constant flow from left to right.

A. In the initial vortex configuration (at the top of figure 4.7), there is a vortex close behind the cylinder and above the flow axis. Velocities are high in the vortex so that pressures are low; both the upward lift force and the drag force (always to the right) will be maximum.

B. In the second image, the vortices have separated from the cylinder a bit. At this instant the pressure field directly along the cylinder will be rather uniform and less pronounced. The lift force will be zero and the drag force will be slightly lower than in the first situation.

C. The third image is a reflection (about the flow axis) of the first. The lift force is now directed downward and is maximum; the drag force is again maximum as well.

D. The fourth image is a reflection of the second one; the lift force is again zero and the drag is again a bit lower.

This pattern keeps on repeating itself. One sees that the lift force oscillates back and forth with a frequency $f_v$; the drag force varies only slightly - a fact that is often neglected completely, by the way - and with a frequency of $2f_v$. This behavior of both the lift and drag will be important for vortex induced vibrations later in this chapter.

While the magnitude of the lift force on a unit length of a cylinder can be of the same order of magnitude as the drag force, one is usually not too concerned about lift forces on offshore structures. This is the case because the drag forces always act in the direction of the current. The elemental drag forces on all cylinder elements add up nicely when integrated over the entire structure to yield a very significant resulting total drag force.
In contrast to this, the lift force direction on each cylinder segment is entirely dependent on the position of the local vortex at that instant. These vortices will be more or less randomly distributed in the wake of the structure at any time. The lift force on one cylinder segment may be acting in one direction, while at that same time the lift force nearby may be acting in the opposite direction. Integration of these elemental forces over the entire structure leads to a resultant which is much smaller than the resultant drag force.

4.6 Vortex Induced Oscillations

There is one condition, however, which can lead to concern about dynamic lift forces: This concern stems from the fact that if the natural frequency of vibration of the structure - think of an umbilical cable to a remote controlled vehicle for example - happens to coincide with the Strouhal frequency at which vortices are shed, then spectacular things can happen. These are discussed here.

To get an idea of some numerical values - assume $D = 1$ m and $U = 1$ m/s - the corresponding Reynolds number is about $1 \cdot 10^6$. For this value of $Rn$ one finds that the Strouhal number, $St$, is about 0.2 so that the vortex shedding period is $T_v = 1/f_v \approx 5$ s.

If the natural oscillation frequency of the structure is $f_n$, then the so-called reduced velocity (a dimensionless value) can be defined as:

$$U_r = \frac{U}{f_n \cdot D} \quad (4.33)$$

The reduced velocity can be seen as an indicator of resonance. Resonance occurs when
the vortex shedding frequency $f_v$ approaches the natural frequency, $f_n$:

$$f_v = \frac{St \cdot U}{D} \rightarrow f_n$$  \hspace{1cm} (4.34)

so that the reduced velocity is then approximately:

$$U_r = \frac{1}{St} \approx 5$$  \hspace{1cm} (4.35)

Thus, once the natural oscillation frequency of a structure (long risers, or umbilical cables are fine examples) is known, one can determine $U_r$ and use it to predict a danger from resonant vortex-induced oscillations.

### 4.6.1 Crosswise Oscillations

To understand what is happening, consider the situation as one slowly increases the flow velocity, $U$, past a cylinder. When the velocity is low enough (but still high enough to cause alternating vortex shedding), the cylinder will remain relatively still in the flow and the vortices will be randomly distributed along the cylinder length in its wake. As $U$ increases, $f_v$ will increase too. As $U_r$ approaches a value of about 5 one observes that the cylinder begins to oscillate significantly in a direction crosswise to the flow. A significant effect of this cylinder oscillation is that it distorts the local flow pattern, making it more attractive for vortices to be shed 'behind' the cylinder (seen from the perspective of both the cylinder motion and the flow). Since each adjacent segment of the cylinder is moving in more or less the same way at any instant in time, this in turn stimulates the vortices to become more coherent along the cylinder length. The more or less randomly distributed lift forces on different cylinder or cable segments now start working in phase with one another. The cable responds to this more coherent lift force excitation by oscillating with a larger amplitude - thus reinforcing the entire process; this is called lock-in.

As the ambient flow velocity continues to increase, lock-in will ultimately stop (usually after $U_r$ reaches a value of about 7) and the cable comes more or less to rest again. People are still trying to explain and accurately predict these phenomena.

### Influence on Drag

A secondary effect of transverse cylinder oscillations can be an increased drag force. There are two "popular and pragmatic" approaches to explain this. Remembering first that the drag force is:

$$F_D = \frac{1}{2} \rho U^2 \cdot C_D \cdot D$$ \hspace{1cm} (4.36)

Then:

- Because the cylinder is oscillating (crosswise to the flow), its wake becomes wider. This has an effect similar to that of increasing $D$; which would increase $F_D$. This is usually expressed by increasing the value of $C_D$, instead.
- Since the cylinder now also moves with speed $dY/dt$, the instantaneous incident flow velocity relative to cylinder becomes:

$$\bar{U}_i = \frac{dY}{dt} \hat{y} + U \hat{x}$$ \hspace{1cm} (4.37)
4.7. SHIP STILL WATER RESISTANCE

So that:

\[ U_i^2 = \left( \frac{dY}{dt} \right)^2 + U^2 \]  \hspace{1cm} (4.38)

This also increases \( F_D \).

Note that in this latter approach the drag force will no longer (strictly speaking) be directed along the \( x \)-axis, parallel to \( U \). Instead, its direction oscillates in phase with \( dY/dt \).

It has been reported that hoisting cables from cranes used to install objects in deep water of at least several hundred meters depth can oscillate in this way. The results have been dramatic at times - with drag coefficients becoming as much as three times the values associated with a fixed wire rope.

The above discussion of forces on an oscillating cylinder is not yet complete. The cylinder oscillation also leads to inertia forces, but these are outside the scope of this chapter; they are discussed in chapter 12.

4.6.2 In-Line Oscillations

The slight time-varying oscillating component of the drag force - superposed on a much more important constant component - has already been mentioned. The amplitude of the "ripple" is usually only a few percent of the total force, and its frequency is twice the Strouhal frequency. Oscillations of the cylinder - if they occur - will now appear for lower values of \( U_r \). Indeed, a cable will sometimes begin a small resonant vibration in the in-line (parallel to the flow) direction when the ambient velocity is only about half that needed to cause a crosswise oscillation.

4.7 Ship Still Water Resistance

This section on the resistance of ships moving in still water is partly based on a text by [Kuiper, 1997] on resistance and propulsion of ships. This text is a good reference for those who wish to know more than is presented here about still water resistance.

The most common purpose of a ship is to transport cargo. The displacement hull form is the most appropriate concept for such ships. The weight of the cargo, stores and fuel is called the deadweight. This deadweight and the weight of the empty ship together are equal to the displacement of the ship. The movement of such a displacement ship requires relatively little power in comparison to other means of transport, at least at low speed. This is because the friction of the water is low and, as long as the waves generated by the moving ship are small, the required amount of power remains small.

A characteristic feature of a displacement ship is that a large amount of cargo is moved at low speeds with low power. As an indication: the power consumption per ton-km cargo transport of a 90 meters length ship at sea is about 1/3 of the power needed for the same transport at land, but the average sustained sea speed of 12 knots is also only about 1/3 of the speed during road transport (assuming there are no excessive traffic jams).

The low speed restriction for a ship is very important; at increasing speeds the required power increases rapidly due to wave generation. At low speeds, the speed-dependent skin friction drag is the major part of the resistance. For an extrapolation from ship model data
to prototype data, this part mainly depends on the Reynolds number. At higher speeds, one observes an increasing radiation of waves by the hull, which means the appearance of a speed-dependent second type of resistance; wave making resistance. For an extrapolation from ship model data to prototype data, this part mainly depends on the Froude number.

In the second part of the 19th century William Froude, 1810 - 1879, (as distinct from his son Robert Edmund Froude from the Froude number) proposed the British Admiralty to build what would become the world’s first towing tank at Torquay in England. He had recently developed scaling laws for predicting the resistance of ships from tests on ship models and he intended to use this tank for the required scale model experiments. The British Admiralty accepted Froude’s proposal on the condition that he also used the tank to investigate the ways of reducing the rolling motion of ships. That was when sails were being replaced by steam driven propulsion systems and roll was becoming more of a problem. On March 3 1872, the first indoor professional model test in the world was carried out in a basin of 85 x 11 x 3 meter, as shown in figure 4.8. About half a century later, Froude’s towing tank was pulled down.

Figure 4.8: The World’s First Towing Tanks

In 1883, the world’s first privately owned towing tank, the Denny tank as shown in figure 4.8 too, was built at Dumbarton in Scotland, using the original design of Froude’s tank in Torquay. The Denny tank - with dimensions 93.00 x 6.80 x 2.75 meter - was closed in 1983, but it was re-opened in 1986 under the umbrella of the Scottish Maritime Museum. The tank testing and museum parts are separate. Since 1989, the tank testing facility operates under the management of Strathclyde University’s Department of Ship and Marine Technology.

It has been the merit of William Froude to distinguish the components of the total hull resistance, \( R_t \), and to relate them to scaling laws; he distinguished between:

- a frictional resistance component, \( R_f \), and
- a residual resistance component, \( R_r \).

Then he made a very drastic simplification, which has worked out remarkably well: Froude’s first hypothesis was that these two components of the resistance are independent of each other. The determination of these two components was a second problem, but he found a simple way out of this problem. Froude’s second hypothesis was that the frictional part
of the resistance can be estimated by the drag of a flat plate with the same wetted area and length as the ship or model. In principle, a flat plate (towed edgewise) has no wave resistance and can therefore be investigated over a range of Reynolds numbers \((Rn = VL/\nu)\) without influence of the wave-related Froude number \((Fn = V/\sqrt{gL})\).

### 4.7.1 Frictional Resistance

To determine the frictional resistance coefficient, William Froude’s approach yielded that the frictional resistance coefficient was related to the resistance coefficient of a flat plate with the same length and wetted surface as the ship or model hull:

\[
C_f = \frac{R_f}{\frac{1}{2}\rho V^2 S} \quad \text{or} \quad R_f = \frac{1}{2}\rho V^2 \cdot C_f \cdot S
\]  

(4.39)

in which:

- \(C_f\) = frictional resistance coefficient (-)
- \(R_f\) = frictional resistance (N)
- \(\rho\) = density of water (kg/m\(^3\))
- \(V\) = ship or model speed (m/s)
- \(S\) = wetted surface of ship or model hull (m\(^2\))

He did numerous experiments to determine the resistance coefficients of a flat plate as a function of the Reynolds number. He himself did not find a single relationship as a function of the Reynolds number due to laminar flow and edge effects in his measurements. His results not only depended on the Reynolds number but also on the length of the plate. Several friction lines based only on the Reynolds number were developed later, both theoretically using boundary layer theory and experimentally.

For lamina\(^\text{r}\) flows, the resistance coefficient was formulated from boundary layer theory by Blasius:

\[
\text{Blasius:} \quad C_f = 1.328 \cdot \sqrt{Rn}
\]  

(4.40)

So-called plate lines were developed for turbulen\(^\text{t}\) boundary layer flows from the leading edge. These plate lines were extended to include full scale Reynolds numbers. They have relatively simple formulations, such as the Schoenherr Mean Line or the ITTC-1957 Line, which are defined as:

\[
\text{Schoenherr:} \quad \frac{0.242}{\sqrt{C_f}} = \log_{10} \left( Rn \cdot C_f \right)
\]  

(4.41)

\[
\text{ITTC-1957:} \quad C_f = \frac{0.075}{\left( \log_{10} (Rn) - 2 \right)^2}
\]  

(4.42)

The latter one is accepted as a standard by the International Towing Tank Conference (ITTC).

As a matter of fact it is not too important that a flat plate with a certain length and wetted surface has a resistance coefficient exactly according to one of the mentioned lines. The Froude hypothesis is crude and correlation factors are required afterwards to arrive at correct extrapolations to full scale values. These correlation factors will depend on the plate line which is used.
4.7.2 Residual Resistance

The determination of the resistance components of a ship’s hull can be illustrated with the results of resistance tests with a series of models at various scales, the “Simon Bolivar” family of models. Resistance tests were carried out over a certain speed range for each of the models. Each model had a different scale factor, $\alpha$. The total resistance (in non-dimensional form) is shown in figure 4.9.

![Figure 4.9: Resistance Coefficients of the “Simon Bolivar” Model Family](source: Kujir 1997)

The **total resistance** force on a ship is made non-dimensional in the same way as the frictional resistance above:

$$ C_t = \frac{R_t}{\frac{1}{2} \rho V^2 S} \quad \text{or} \quad R_t = \frac{1}{2} \rho V^2 \cdot C_t \cdot S $$

(4.43)

in which:

- $C_t$ = total resistance coefficient (-)
- $R_t$ = total resistance (N)
- $\rho$ = density of water (kg/m$^3$)
- $V$ = ship or model speed (m/s)
- $S$ = wetted surface of ship or model hull (m$^2$)
4.7. SHIP STILL WATER RESISTANCE

$C_t$ is plotted as a function of the logarithm of the Reynolds number, $R_n = V \cdot L_{WL}/\nu$, in which $L_{WL}$ is the length at the water plane. Each curve of data points has been made with a model of the given scale. The Schoenherr Mean Line follows from equation 4.41; the other two curves connect the points for $F_n$ is zero and for $F_n$ is another constant, respectively.

Similar lines can be drawn for other Froude numbers. Such a line of constant Froude number is approximately parallel to this plate line. If all scaling laws could be satisfied, the resistance curves at all model scales would coincide. Because the Reynolds number is not maintained, this is not the case and each model has a separate curve.

The residual resistance coefficient, $C_r$, at a certain Froude number is now the vertical distance between the plate line and the line for that Froude number. When the plate line and the line of constant Froude number are parallel, this means that the residual resistance component is indeed independent of the Reynolds number. This is assumed to be always the case in Froude’s method and the residual resistance at each Froude number is determined by subtracting the calculated frictional resistance coefficient of a flat plate according to equations 4.41 or 4.42 from the measured total resistance.

The assumption that the residual resistance is approximately independent of the Reynolds number is not accurate in many cases. This means that the lines of constant Froude number in figure 4.9 are not exactly parallel. The residual resistance in that case contains a component which depends on the Reynolds number.

To determine this component, the line for $F_n \to 0$ has been determined in figure 4.9. When the Froude number becomes sufficiently small the generated waves will disappear and the difference between the total resistance coefficient and the plate line can only be associated with the longer distance of water particles to travel along the (not flat) hull form and some small viscous effects. Thus the difference between the plate line and the line $F_n \to 0$ has to do with the form of the hull and is therefore called form resistance. The line $F_n \to 0$ is now assumed to be always parallel to the lines for other Froude numbers, so that the difference between the total resistance coefficient and the line $F_n \to 0$ is independent of the Reynolds number. This difference is now called wave resistance.

In this way the total resistance is split up into:

- the frictional resistance (from the plate line),
- the form resistance and
- the wave resistance.

Form resistance and wave resistance together constitute Froude’s residual resistance.

Form Resistance.

Several hypotheses have been made about the relation between the form resistance and the frictional resistance. As already mentioned, Froude assumed that the form resistance was independent of the Reynolds number. He therefore included it as a part of the residual resistance. Hughes in 1953 assumed that the form drag was proportional to the viscous resistance and multiplied the viscous resistance coefficient, $C_f$, by a constant factor, $k$, as shown in figure 4.10.

The Reynolds-dependent component of the resistance thus becomes $(1+k) \cdot C_f$. The factor $(1+k)$ is called the form factor. Froude’s method is still used, but Hughes’ approach of is the most widely adopted one and is the accepted standard by the ITTC.
Figure 4.10: Hughes Extrapolation Method

The form factor is influenced primarily by the shape of the after part of the ship but it is often given as a function of the block coefficient, $C_B$, of the ship as a whole. The block coefficient is the ratio between the ship’s displacement volume and that of a rectangular box in which the ship’s underwater volume "just fit". Form factors associated with typical block coefficients are listed below.

<table>
<thead>
<tr>
<th>$C_B$</th>
<th>$1 + k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;0.7</td>
<td>1.10-1.15</td>
</tr>
<tr>
<td>0.7-0.8</td>
<td>1.15-1.20</td>
</tr>
<tr>
<td>&gt;0.8</td>
<td>1.20-1.30</td>
</tr>
</tbody>
</table>

Wave Resistance.

The wave resistance coefficient of the model, $C_w$, over the speed range of the model is now found by subtracting the frictional and form resistance coefficients from the measured total resistance coefficient:

$$C_w = C_t - (1 + k) \cdot C_f$$  \hspace{1cm} (4.44)

4.7.3 Extrapolation of Resistance Tests

Given the components of the total resistance of the model, one must extrapolate this data to full scale. The resistance of the model is generally measured from a low speed up to the design speed. The model design speed is set by maintaining the full scale Froude number. Equation 4.43 is used to express the total resistance in dimensionless form.

The wetted surface, $S$, is taken as the wetted length from keel to still water line of the frames, integrated over the ship length and multiplied by two, to account for both sides
4.8. WIND LOADS

of the ship. Note that the ship length is measured directly along the center line and not along the water line.

The Froude number is maintained during the model test. This means that the wave resistance coefficient, $C_w$, at model scale and at full scale are the same. The total resistance coefficient of the ship, $C_{t_{ship}}$, can therefore be found from:

$$C_{t_{ship}} = (1 + k) \cdot C_{f_{ship}} + C_w + C_a$$

(4.45)

The form factor, $k$, and the wave resistance coefficient, $C_w$, are found directly from the model test. The frictional resistance coefficient at full scale, $C_{f_{ship}}$, can be read from the plate line using the full scale Reynolds number.

The additional resistance coefficient, $C_a$, is a new element. This coefficient is a correlation coefficient based on experience at full scale. It accounts both for extrapolation errors due to the various assumptions made and for effects at full scale which are not present at the model. Such effects include the relatively rough surface at full scale in relation to the boundary layer thickness. Moreover, $C_a$ contains a correction for the difference in air resistance of the above water part of the hull of model and prototype. [Holtrop and Mennen, 1982] give a simple formula for $C_a$ as a fraction of the length of the ship:

$$C_a = 0.006 \cdot (L_{WL} + 100)^{-0.16} - 0.00205$$

(4.46)

in which $L_{WL}$ is the length of the water plane in meters.

The additional resistance coefficient decreases in this formula with increasing length and this can be attributed, at least partly, to roughness effects. The effect of surface roughness in general requires special attention.

4.7.4 Resistance Prediction Methods

Thus, the total resistance of the ship follows from:

$$R_{t_{ship}} = \frac{1}{2} \rho V^2 \cdot C_{t_{ship}} \cdot S$$

(4.47)

and the resistance coefficient, $C_{ts}$, has to be determined.

A number of methods to determine the still water resistance coefficients of ships, based on (systematic series of) model test data, are given in the literature. A very well known method, developed at MARIN, is described by [Holtrop, 1977], [Holtrop and Mennen, 1982] and [Holtrop, 1984]. The method is based on the results of resistance tests carried out by MARIN during a large number of years and is available in a computerized format. The reader is referred to these reports for a detailed description of this method, often indicated by the "Holtrop and Mennen" method. An example for a tug of the correlation between the results of this "Holtrop and Mennen" method and model test results is given in figure 4.11.

4.8 Wind Loads

Like all environmental phenomena, wind has a stochastic nature which greatly depends on time and location. It is usually characterized by fairly large fluctuations in velocity and
direction. It is common meteorological practice to give the wind velocity in terms of the average over a certain interval of time, varying from 1 to 60 minutes or more.

Local winds are generally defined in terms of the average velocity and average direction at a standard height of 10 meters above the still water level. A number of empirical and theoretical formulas are available in the literature to determine the wind velocity at other elevations. An adequate vertical distribution of the true wind speed \( z \) meters above sea level is represented by:

\[
\frac{V_{tw}(z)}{V_{tw}(10)} = \left( \frac{z}{10} \right)^{0.11} \quad \text{(at sea)}
\]  

(4.48)

in which:

\[ V_{tw}(z) = \text{true wind speed at } z \text{ meters height above the water surface} \]
\[ V_{tw}(10) = \text{true wind speed at 10 meters height above the water surface} \]

Equation 4.48 is for sea conditions and results from the fact that the sea is surprisingly smooth from an aerodynamic point of view - about like a well mowed soccer field. On land, equation 4.48 has a different exponent:

\[
\frac{V_{tw}(z)}{V_{tw}(10)} = \left( \frac{z}{10} \right)^{0.16} \quad \text{(on land)}
\]  

(4.49)

At sea, the variation in the mean wind velocity is small compared to the wave period. The fluctuations around the mean wind speed will impose dynamic forces on an offshore structure, but in general these aerodynamic forces may be neglected in comparison with the hydrodynamic forces, when considering the structures dynamic behavior. The wind will be considered as steady, both in magnitude and direction, resulting in constant forces and a constant moment on a fixed floating or a sailing body.

The wind plays two roles in the behavior of a floating body:

---

Figure 4.11: Comparison of Resistance Prediction with Model Test Results
4.8. WIND LOADS

- Its first is a direct role, where the wind exerts a force on the part of the structure exposed to the air. Wind forces are exerted due to the flow of air around the various parts. Only local winds are needed for the determination of these forces.

- The second is an indirect role. Winds generate waves and currents and through these influence a ship indirectly too. To determine these wind effects, one needs information about the wind and storm conditions in a much larger area. Wave and current generation is a topic for oceanographers; the effects of waves and currents on floating bodies will be dealt with separately in later chapters.

Only the direct influence of the winds will be discussed here. Forces and moments will be caused by the speed of the wind relative to the (moving) body. The forces and moments which the wind exerts on a structure can therefore be computed by:

\[
\begin{align*}
X_w &= \frac{1}{2} \rho_{\text{air}} V_{rw}^2 \cdot C_{Xw}(\alpha_{rw}) \cdot A_T \\
Y_w &= \frac{1}{2} \rho_{\text{air}} V_{rw}^2 \cdot C_{Yw}(\alpha_{rw}) \cdot A_L \\
N_w &= \frac{1}{2} \rho_{\text{air}} V_{rw}^2 \cdot C_{Nw}(\alpha_{rw}) \cdot A_L \cdot L
\end{align*}
\]

in which:

- \(X_w\) = steady longitudinal wind force (N)
- \(Y_w\) = steady lateral wind force (N)
- \(N_w\) = steady horizontal wind moment (Nm)
- \(\rho_{\text{air}} \approx \rho_{\text{water}}/800\) = density of air (kg/m\(^3\))
- \(V_{rw}\) = relative wind velocity (m/s)
- \(\alpha_{rw}\) = relative wind direction (-), from astern is zero
- \(A_T\) = transverse projected wind area (m\(^2\))
- \(A_L\) = lateral projected wind area (m\(^2\))
- \(L\) = length of the ship (m)
- \(C_{sw}(\alpha_{rw})\) = \(\alpha_{rw}\)-dependent wind load coefficient (-)

Note that it is a “normal” convention to refer to the true wind direction as the direction from which the wind comes, while waves and currents are usually referred to in terms of where they are going. A North-West wind will cause South-East waves, therefore!

4.8.1 Wind Loads on Moored Ships

For moored ships, only the true wind speed and direction determine the longitudinal and lateral forces and the yaw moment on the ship, as given in figure 4.12. Because of the absence of a steady velocity of the structure, the relative wind is similar to the true wind:

\[
|V_{rw} = V_{tw}| \quad \text{and} \quad |\alpha_{rw} = \alpha_{tw}|
\]

The total force and moment experienced by an object exposed to the wind is partly of viscous origin (pressure drag) and partly due to potential effects (lift force). For blunt
bodies, the wind force is regarded as independent of the Reynolds number and proportional to the square of the wind velocity.

[Remery and van Oortmerssen, 1973] collected the wind data on 11 various tanker hulls. Their wind force and moment coefficients were expanded in Fourier series as a function of the angle of incidence. From the harmonic analysis, it was found that a fifth order representation of the wind data is sufficiently accurate, at least for preliminary design purposes:

\[
C_{Xw} = a_0 + \sum_{n=1}^{5} a_n \sin(n \cdot \alpha_{rw}) \\
C_{Yw} = \sum_{n=1}^{5} b_n \sin(n \cdot \alpha_{rw}) \\
C_{Nw} = \sum_{n=1}^{5} c_n \sin(n \cdot \alpha_{rw})
\]  

(4.52)

with wind coefficients as listed below.

<table>
<thead>
<tr>
<th>Tanker No.</th>
<th>Length Lpp</th>
<th>Condition</th>
<th>Bridge Location</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>loaded</td>
<td>at (\frac{L}{2})</td>
<td>ballast</td>
<td>at (\frac{L}{2})</td>
<td>aft</td>
<td>ballast</td>
<td>loaded</td>
<td>at (\frac{L}{2})</td>
<td>aft</td>
<td>ballast</td>
<td>loaded</td>
<td>at (\frac{L}{2})</td>
<td>aft</td>
</tr>
<tr>
<td>a_0</td>
<td></td>
<td>-0.131</td>
<td>-0.079</td>
<td>0.028</td>
<td>0.014</td>
<td>-0.074</td>
<td>-0.055</td>
<td>-0.038</td>
<td>-0.039</td>
<td>-0.042</td>
<td>-0.075</td>
<td>-0.051</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a_1</td>
<td></td>
<td>0.748</td>
<td>0.615</td>
<td>0.799</td>
<td>0.732</td>
<td>1.030</td>
<td>0.748</td>
<td>0.830</td>
<td>0.646</td>
<td>0.487</td>
<td>0.711</td>
<td>0.577</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a_2</td>
<td></td>
<td>0.059</td>
<td>0.095</td>
<td>-0.054</td>
<td>-0.017</td>
<td>-0.062</td>
<td>-0.012</td>
<td>0.012</td>
<td>0.024</td>
<td>0.109</td>
<td>0.043</td>
<td>0.034</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a_3</td>
<td></td>
<td>0.108</td>
<td>0.076</td>
<td>0.018</td>
<td>-0.018</td>
<td>0.086</td>
<td>0.015</td>
<td>0.021</td>
<td>-0.031</td>
<td>0.075</td>
<td>0.064</td>
<td>0.062</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a_4</td>
<td></td>
<td>-0.001</td>
<td>0.025</td>
<td>-0.018</td>
<td>-0.058</td>
<td>-0.110</td>
<td>-0.151</td>
<td>-0.072</td>
<td>-0.090</td>
<td>-0.047</td>
<td>-0.038</td>
<td>0.006</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a_5</td>
<td></td>
<td>0.786</td>
<td>0.880</td>
<td>0.697</td>
<td>0.785</td>
<td>0.707</td>
<td>0.731</td>
<td>0.718</td>
<td>0.735</td>
<td>0.764</td>
<td>0.819</td>
<td>0.879</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b_1</td>
<td></td>
<td>0.009</td>
<td>0.004</td>
<td>0.036</td>
<td>0.014</td>
<td>-0.013</td>
<td>-0.014</td>
<td>0.032</td>
<td>0.003</td>
<td>0.037</td>
<td>0.031</td>
<td>0.026</td>
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</tr>
<tr>
<td>b_2</td>
<td></td>
<td>0.003</td>
<td>0.003</td>
<td>0.018</td>
<td>0.014</td>
<td>0.029</td>
<td>0.016</td>
<td>0.010</td>
<td>0.004</td>
<td>0.052</td>
<td>0.023</td>
<td>0.014</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b_3</td>
<td></td>
<td>0.004</td>
<td>0.004</td>
<td>0.028</td>
<td>0.015</td>
<td>0.007</td>
<td>0.001</td>
<td>0.001</td>
<td>-0.005</td>
<td>0.016</td>
<td>0.032</td>
<td>0.031</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b_4</td>
<td></td>
<td>-0.019</td>
<td>-0.003</td>
<td>-0.023</td>
<td>-0.029</td>
<td>-0.044</td>
<td>-0.025</td>
<td>-0.040</td>
<td>-0.017</td>
<td>-0.003</td>
<td>-0.032</td>
<td>0.029</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b_5</td>
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<td>0.761</td>
<td>0.096</td>
<td>-0.371</td>
<td>-0.074</td>
<td>0.006</td>
<td>-0.010</td>
<td>0.075</td>
<td>0.064</td>
<td>0.819</td>
<td>0.879</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 - c_1</td>
<td></td>
<td>-0.451</td>
<td>-0.338</td>
<td>-0.765</td>
<td>-0.524</td>
<td>-0.236</td>
<td>-0.059</td>
<td>-0.526</td>
<td>-0.315</td>
<td>-0.025</td>
<td>-0.081</td>
<td>-0.644</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 - c_2</td>
<td></td>
<td>-0.617</td>
<td>-0.800</td>
<td>-0.574</td>
<td>-0.738</td>
<td>-0.531</td>
<td>-0.730</td>
<td>-0.056</td>
<td>-0.722</td>
<td>-0.721</td>
<td>-0.681</td>
<td>-0.726</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 - c_3</td>
<td></td>
<td>-0.110</td>
<td>-0.090</td>
<td>-0.166</td>
<td>-0.175</td>
<td>-0.063</td>
<td>-0.035</td>
<td>-0.111</td>
<td>-0.090</td>
<td>-0.345</td>
<td>-0.202</td>
<td>-0.244</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 - c_4</td>
<td></td>
<td>-0.110</td>
<td>-0.096</td>
<td>-0.146</td>
<td>-0.080</td>
<td>-0.073</td>
<td>-0.017</td>
<td>-0.113</td>
<td>-0.047</td>
<td>-0.127</td>
<td>-0.145</td>
<td>-0.076</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 - c_5</td>
<td></td>
<td>-0.010</td>
<td>-0.013</td>
<td>0.021</td>
<td>-0.021</td>
<td>0.024</td>
<td>-0.013</td>
<td>0.099</td>
<td>0.067</td>
<td>-0.022</td>
<td>0.039</td>
<td>0.024</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.13 shows, as an example, the measured wind forces and moment together with their Fourier approximation, for one of the tankers.
4.8. WIND LOADS

4.8.2 Wind Loads on Other Moored Structures

The wind forces on other types of structures, as for instance semi-submersible platforms, can be approximated by dividing the structure into a number of components, all with a more or less elementary geometry, and estimating the wind force on each element. Drag coefficients are given in the literature for a lot of simple geometrical forms, such as spheres, flat plates and cylinders of various cross sectional shapes. [Hoerner, 1965] and [Delany and Sorensen, 1970] are good sources of this information. The total wind load on the structure is found by adding the contributions of all the individual component parts. The fact that one element may influence the wind field of another element is neglected in this analysis.

4.8.3 Wind Loads on Sailing Ships

For sailing merchant ships and tankers, only the longitudinal wind resistance, \( X_w = R_w \), is of importance for determining a sustained sea speed.

The relative wind speed, \( V_{rw} \), and the relative wind direction, \( \alpha_{rw} \), have to be determined from this true wind speed, \( V_{tw} \), and the true wind direction, \( \alpha_{tw} \), together with the forward ship speed, \( V_s \), and the heading of the ship, see figure 4.14.

As opposed to the hydromechanical notation, for seamen head wind has a direction equal to zero. The effect of the lateral force and the yaw moment on the ship will be corrected via a small course correction. Then the relative wind speed and the relative wind direction (so head wind has a direction equal to zero) follows from:

\[
V_{rw} = \sqrt{V_s^2 + V_{tw}^2 + 2 \cdot V_s \cdot V_{tw}} \\
\alpha_{rw} = \arctan \left( \frac{V_{tw} \cdot \sin \alpha_{tw}}{V_s + V_{tw} \cdot \cos \alpha_{tw}} \right)
\]

(4.53)
[Isherwood, 1973] published a reliable method for estimating the wind resistance. He analyzed the results of wind resistance experiments carried out at different laboratories with models covering a wide range of merchant ships. He determined empirical formulas for the two horizontal components of the wind force and the wind-induced horizontal moment on any merchant ship form for a wind coming from any direction.

The longitudinal wind resistance is defined by:

\[
X_w = \frac{1}{2} \rho_{air} V_{rw}^2 \cdot C_{X_w} \cdot A_T
\]  

(4.54)

with for the longitudinal wind resistance coefficient \(C_{X_w}\):

\[
C_{X_w} = A_0 + A_1 \cdot \frac{2A_L}{L_{\text{oa}}^2} + A_2 \cdot \frac{2A_T}{B^2} + A_3 \cdot \frac{L_{\text{oa}}}{B} + A_4 \cdot \frac{S}{L_{\text{oa}}} + A_5 \cdot \frac{C}{L_{\text{oa}}} + A_6 \cdot M
\]  

(4.55)

in which:

- \(L_{\text{oa}}\) = length over all (m)
- \(B\) = beam (m)
- \(S\) = length of perimeter of lateral projection excluding water line and slender bodies such as masts and ventilators (m)
- \(C\) = distance from bow of centroid of lateral projected area (m)
- \(A_L\) = lateral projected wind area (m²)
- \(A_T\) = transverse projected wind area (m²)
- \(M\) = number of distinct groups of masts or kingposts seen in lateral projection (-); kingposts close against the bridge are not included

The coefficients for equation 4.55 are listed below.
4.9 Current Loads

There are several independent phenomena responsible for the occurrence of current: the ocean circulation system resulting in a steady current, the cyclical change in lunar and solar gravity causing tidal currents, wind and differences in sea water density. The steady wind velocity at the water surface is about 3 per cent of the wind velocity at 10 meters height. Tidal currents are of primary importance in areas of restricted water depth and can attain values up to 10 knots. However, such extreme velocities are rare; a 2-3 knots tidal current speed is common in restricted seas. The prediction of tidal currents is left for the oceanographers.

Although surface currents will be the governing ones for floating structures; the current distribution as a function of depth below the surface may also be of importance. For the design of a mooring system of a floating structure, the designer is especially interested in the probability that a particular extreme current velocity will be exceeded during a certain period of time. Observations obtained from current speed measurements are indispensable for this purpose. It may be useful to split up the total measured current in two or more components, for instance in a tidal and a non-tidal component, since the direction of the various components will be different, in general. The variation in velocity and direction of the current is very slow, and current may therefore be considered as a steady phenomenon.

The forces and moment exerted by a current on a floating object is composed of the following parts:

- A viscous part, due to friction between the structure and the fluid, and due to pressure drag. For blunt bodies the frictional force may be neglected, since it is small compared to the viscous pressure drag.

- A potential part, with a component due to a circulation around the object, and one from the free water surface wave resistance. In most cases, the latter component is small in comparison with the first and will be ignored.
The forces and moments, as given in figure 4.12, exerted by the current on a floating structure can be calculated from:

\[
\begin{align*}
X_c &= \frac{1}{2} \rho \cdot V_c^2 \cdot C_{Xc}(\alpha_c) \cdot A_{TS} \\
Y_c &= \frac{1}{2} \rho \cdot V_c^2 \cdot C_{Yc}(\alpha_c) \cdot A_{LS} \\
N_c &= \frac{1}{2} \rho \cdot V_c^2 \cdot C_{Nc}(\alpha_c) \cdot A_{LS} \cdot L
\end{align*}
\]

in which:

- \(X_c\) = steady longitudinal current force (N)
- \(Y_c\) = steady lateral current force (N)
- \(N_c\) = steady yaw current moment (Nm)
- \(\rho\) = density of water (kg/m\(^3\))
- \(V_c\) = current velocity (m/s)
- \(\alpha_c\) = current direction, from astern is zero (rad)
- \(A_{TS} \approx B \cdot T\) = submerged transverse projected area (m\(^2\))
- \(A_{LS} \approx L \cdot T\) = submerged lateral projected area (m\(^2\))
- \(L\) = length of the ship (m)
- \(B\) = breadth of the ship (m)
- \(T\) = draft of the ship (m)
- \(C_{-c}(\alpha_c)\) = \(\alpha_c\)-depending current load coefficient (-)

Results of model tests are given in the literature for various types of structures and vessels.

### 4.9.1 Current Loads on Moored Tankers

[Remery and van Oortmerssen, 1973] published current loads on several tanker models of different sizes, tested at MARIN. The coefficients \(C_{Xc}\), \(C_{Yc}\) and \(C_{Nc}\) were calculated from these results. A tanker hull is a rather slender body for a flow in the longitudinal direction and consequently the longitudinal force is mainly frictional. The total longitudinal force was very small for relatively low current speeds and could not be measured accurately. Moreover, extrapolation to full scale dimensions is difficult, since the longitudinal force is affected by scale effects.

For mooring problems the longitudinal force will hardly be of importance. An estimate of its magnitude can be made by calculating the flat plate frictional resistance, according to the ITTC skin friction line as given in equation 4.42:

\[
X_c = \frac{0.075}{\left(\log_{10} (Rn) - 2\right)^2} \cdot \frac{1}{2} \rho V_c^2 \cdot \cos \alpha_c \cdot \left| \cos \alpha_c \right| \cdot S
\]

while:

\[
Rn = \frac{V_c \cdot \left| \cos \alpha_c \right| \cdot L}{\nu}
\]
4.9. **CURRENT LOADS**

\[
S \approx L \cdot (B + 2T) = \text{wetted surface of the ship (m}^2) \\
L = \text{length of the ship (m)} \\
B = \text{breadth of the ship (m)} \\
T = \text{draft of the ship (m)} \\
V_c = \text{current velocity (m/s)} \\
\alpha_c = \text{current direction (-), from astern is zero} \\
\rho = \text{density of water (ton/m}^3) \\
Rn = \text{Reynolds number (-)} \\
\nu = \text{kinematic viscosity of water (m}^2\text{s)}
\]

Extrapolation of the transverse force and yaw moment to prototype values is no problem. For flow in the transverse direction a tanker is a blunt body and, since the bilge radius is small, flow separation occurs in the model in the same way as in the prototype. Therefore, the transverse force coefficient and the yaw moment coefficient are independent of the Reynolds number.

The coefficients for the transverse force and the yaw moment were expanded by MARIN in a Fourier series, as was done for the wind load coefficients as described in a previous section:

\[
C_{Yc}(\alpha_c) = \sum_{n=1}^{5} b_n \cdot \sin(n \cdot \alpha_c) \\
C_{Nc}(\alpha_c) = \sum_{n=1}^{5} c_n \cdot \sin(n \cdot \alpha_c)
\]

(4.59)

The average values of the coefficients \(b_n\) and \(c_n\) for the fifth order Fourier series, as published by [Remery and van Oortmerssen, 1973], are given in the table below.

<table>
<thead>
<tr>
<th>n</th>
<th>(b_n)</th>
<th>(10 \cdot c_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.908</td>
<td>-0.252</td>
</tr>
<tr>
<td>2</td>
<td>0.000</td>
<td>-0.904</td>
</tr>
<tr>
<td>3</td>
<td>-0.116</td>
<td>0.032</td>
</tr>
<tr>
<td>4</td>
<td>0.000</td>
<td>0.109</td>
</tr>
<tr>
<td>5</td>
<td>-0.011</td>
<td>0.011</td>
</tr>
</tbody>
</table>

These results are valid for deep water. For shallow water, the transverse current force and moment coefficients have to be multiplied by a coefficient, which is given in figure 4.15.

The influence of the free surface is included in the data given on the coefficients \(b_n\) and \(c_n\) in the previous table. This influence, however, depends on the water depth and on the Froude number, and consequently changes if the current velocity or the tanker dimensions change. For the condition to which these data apply, deep water and a prototype current speed in the order of 3 knots, the effect of the free surface is very small. For the case of a small clearance under the keel and a current direction of 90 degrees, damming up of the water at the weather side and a lowering of the water at the lee side of the ship occurs.

### 4.9.2 Current Loads on Other Moored Structures

Current loads on other types of floating structures are usually estimated in the same way as is used for wind loads.
4.9.3 Current Loads on Sailing Ships

For sailing ships, generally it is assumed that the ship is moving with the current. So the forward ship speed is not the ground speed but the speed relative to the water. Then there is no current load; the current itself is a navigation problem only.

4.10 Thrust and Propulsion

Now that the resistance of a ship or other floating objects moving through still water has been discussed, it is appropriate to approach this phenomena from the other side by discussing the propulsion systems needed to overcome the resistance. This section on that topic is partly based on a text by [Kuiper, 1997] on resistance and propulsion of ships. This text is a good reference for those who wish to know more than is presented here about thrust and propulsion.

The basic action of propulsors like propellers is to deliver thrust. In fact, a propulsor is an energy transformer, because torque and rotation, delivered to the propulsor, will be transformed into thrust and translation, delivered by the propulsor. A consequence is that the propulsor also generates water velocities in its wake, which represent a loss of kinetic energy. It is obvious that this will effect the efficiency of the propulsor, defined by:

\[
\eta = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{P_E}{P_D} = \frac{T \cdot V_e}{Q \cdot 2\pi n}
\]

in which:
4.10. THRUST AND PROPULSION

\[ \eta = \text{propulsive efficiency} \quad (-) \]
\[ P_D = \text{delivered power}, \text{delivered to the propulsor (Nm/s = W)} \]
\[ P_E = \text{effective power}, \text{delivered by the propulsor (Nm/s = W)} \]
\[ Q = \text{torque delivered to the propulsor (Nm)} \]
\[ n = \text{number of revolutions (1/s)} \]
\[ T = \text{thrust delivered by the propulsor (N)} \]
\[ V_e = \text{mean entrance speed of water in propeller disk, also called advance velocity with notation } V_a \text{ (m/s)} \]

The efficiency varies widely between various types of propulsors, but the **screw propeller** has not yet been equalled in most cases and is therefore the most commonly used propulsor. The propulsor is generally mounted behind the hull. This is because of efficiency; the water which is brought into motion by the friction along the ship is reversed by the propeller action and as a result less energy is left behind in the water.

A risk for every propulsor operating at high rotational velocities is **cavitation**. This occurs when the local pressure, associated with high local velocities in the fluid, is lower than the vapor pressure. As a result the water starts to boil locally and water-vapour filled cavities are formed. When these vapor-filled (not air-filled) cavities in the wake arrive in regions with a higher pressure they collapse violently, causing local shock waves in the water that can **erode** the nearby surface. This dynamic behavior of large cavities can also generate **vibrations** in the ship structure.

4.10.1 Propulsors

The most important propulsors used for shipping and offshore activities include:

- **Fixed Pitch Propellers**, see figure 4.16-a.
  The most common propulsor is the fixed pitch open screw propeller (FPP) which, as all propellers, generates a propulsive force by lift on the propeller blades. These blade sections are similar to those of airfoils, operating at some angle of attack in the flow. The geometry of the propeller blades is quite critical with respect to the occurrence of cavitation. Therefore, a specific propeller is generally designed for the specific circumstances of each ship and its engine. The thrust, and consequently the speed of the ship, is controlled by the propeller rotational speed - often called revolutions or rpm (for revolutions per minute).

- **Controllable Pitch Propellers**, see figure 4.16-b.
  In the case of a controllable pitch propeller (CPP) the thrust is controlled by changing the pitch of the blades. The shaft often has a constant rotational speed. A CCP is often used when the propeller has to operate at more than one design condition - such as free running and towing for a tug. It is also effective when rapid maneuvering is required or when shaft generators, which require a constant rpm, are present. Reversing the thrust occurs by changing the pitch with constant revolutions in the same direction. This significantly decreases the time required to change the direction of the thrust. It is important to keep in mind that the CPP has only one design pitch; changing the pitch always reduces the efficiency.

- **Ducted Propellers**, see figure 4.16-c.
  At high propeller loadings a duct increases the propeller efficiency. A duct generates
part of the total thrust due to its interaction with the propeller. This is the case with an accelerating duct, in which the flow velocity is increased due to the duct. Ducted propellers are used in a wide range of applications with heavily loaded propellers, such as for tugs. Several types of ducts, sometimes asymmetric and sometimes at some distance upstream of the propeller, have been designed to make the propeller inflow more uniform.

- **Thrusters**, see figures 4.16-d and 4.16-e.

  A propeller can be driven from above by a vertical shaft. This makes it possible to rotate the propeller along the vertical axis and to generate thrust in any chosen direction. These configurations are called thrusters. They can have an open propeller, but very often a duct is also used. The right angle drive of a thruster makes it more complicated and thus more expensive and vulnerable than a normal propeller shaft. Also the hub diameter is larger, which slightly decreases efficiency. There are some interesting advantages however. The propeller can be in front of the vertical
4.10. THRUST AND PROPULSION

shaft (towing) as well as behind it (pushing). In the towing mode the inflow is more uniform; this decreases vibrations and cavitation. A steerable or azimuthing thruster may rotate around a vertical axis which makes it possible to steer with it. This makes it easier to manoeuver, especially at low speeds. They are common for dynamic positioning; the steerable direction of its thrust is fully utilized in that case.

- **Cycloidal or Voith-Schneider Propellers**, see figures 4.16-f and 4.16-h.
  A very special propulsor is the cycloidal propeller, also called Voith-Schneider propeller after its main developer. It consists of a number of foils on a rotating plate. These foils can rotate relative to this plate and their position is such that they are always perpendicular to the radii from a moving center point, $P$, as shown in figure 4.16-h. When this center point is in the center of the blade circle, there is no resulting force. When this center point is moved, a thrust is generated perpendicular to the direction in which the center point is shifted. Thus thrust can be applied in any direction just by moving the center point; rudders can be omitted. This propulsive system can be used for tugs and supply boats, for which maneuvering is important. Its efficiency, however, is lower than that of an open propeller due to the fact that the blades generate thrust over a part of their revolution only, while viscous resistance is present over the whole revolution. Voith-Schneider propellers must be mounted under a flat bottom; a bottom cover is sometimes provided for protection (see figure 4.16-f).

- **Water Jets**, see figure 4.16-g.
  This propulsor accelerates water using a pump inside the hull, instead of a propeller outside the hull. The water is drawn in from the bottom of the ship, is accelerated inside the ship by a pump and leaves the ship at the stern. This has many advantages when a propeller is too vulnerable to damage, or when a propeller is too dangerous as is the case for rescue vessels. Rudders can be omitted because of the rotating possibilities of the outlet and excellent maneuvering qualities can be obtained, as for instance are required for pilot vessels. A pump jet can be useful in shallow water. However, the inner surface of the pump system is large and the velocities inside are high; the viscous losses are high, too. The efficiency is therefore lower than that of an open propeller.

4.10.2 Propeller Geometry

First some general geometrical definitions of the propeller are given. A sketch of a right handed propeller is given in figure 4.17-a.

- The propeller blades are attached to the hub, which is attached to the end of the propeller shaft. The propeller rotates about the shaft center line. The direction of rotation is as viewed from behind, that is towards the shaft. In normal forward operation a right handed propeller rotates in clockwise direction when viewed from behind.
  The propeller hub is, of course, rotational symmetric because it should not disturb the flow. The attachment of the propeller blade to the hub is a gradual transition, which is done in the fillet area or the blade root. A streamlined cap is generally fitted to the hub at the end of its shaft.
The front edge of the blade is called the leading edge or nose and the other end is the trailing edge or tail. The outermost position, where leading and trailing edges meet, is called the blade tip. The radius of the tip is called the propeller radius and the propeller diameter is, of course, twice its radius.

The surface of the blade which is at the side from which the shaft comes is called the propeller back and the other side is the face of the propeller; when the ship moves forward, the propeller inflow is from its back. Because in forward speed the back side has a low average pressure and the face side has a high average pressure (this pressure difference generates the thrust), the face is also called the pressure side and the back the suction side.

Consider now an arbitrary propeller as drawn in figure 4.18-a.
The intersection of a cylinder with radius \( r \) and a propeller blade, the \textit{blade section}, in figure 4.18-a has the shape of an airfoil. Such a shape is also called just a \textit{foil} or a \textit{profile}, as given in figure 4.18-b.

The straight line between the leading and trailing edge of the profile is called the \textit{chord line} of the profile and the distance between nose and tail is the \textit{chord length}, \( c \), of the foil.

Generally, the origin of the local coordinate system of a profile is taken at the leading edge. The \( x \)-direction is towards the tail, the \( y \)-direction upwards, perpendicular to the chord. The angle between the nose-tail line and the undisturbed flow (relative to the blade) is the \textit{angle of attack}, \( \alpha \).

The distance between the suction side and the pressure side, measured perpendicular to the chord is the \textit{thickness}, \( t(x) \), of the profile. The line of mid-thickness over the chord is the \textit{camber line}.

The blade sections of the propeller have a certain pitch. The chord line or nose-tail line of the blade section - a helix on the cylinder - becomes a straight pitch line, if the cylinder is developed on to a flat surface. The \textit{propeller pitch}, \( P \), is defined as the increase in axial direction of the pitch line over one full revolution \( 2\pi r \) at each radius \( r \); see figure 4.19-a. The dimension of the pitch is a length. The ratio \( P/D \) is the \textit{pitch ratio}. The \textit{pitch angle}, \( \theta = \arctan (P/2\pi r) \), is the angle between the pitch line and a plane perpendicular to the propeller shaft. The pitch distribution is given in a \textit{pitch diagram}, which is simply a graph of the pitch at every radius.

Figure 4.19-b shows the axial velocity \( V_e \) (speed of entrance) and the rotational velocity \( 2\pi nr \) of the water particles at a radius \( r \) from the propeller axis. As a propeller is rotating in water, it can not advance \( P \cdot n \) and a certain difference occurs. The difference between \( P \cdot n \) and \( V_e \) is called the \textit{slip} of the propeller.

A significant radius, which is often used as representative for the propeller, is the radius at \( r/R = 0.7 \). If a pitch is given in the case of a variable pitch distribution it is usually the pitch at \( 0.7R \). Note that half the area of the propeller disk is within a circle with radius \( 0.7R \) and that, consequently, the other half is outside this region;
so the pressure at this circular line is "more or less" the average pressure over the 
full propeller disk.

Now, some important contours and areas will be defined:

- A plane perpendicular to the shaft and through the middle of the chord of the root 
  section is called the **propeller plane**, see figure 4.17-b.

- The projection of the blade contour on the propeller plane gives the **projected blade 
  contour** and its area is called the **projected blade area**. The blade sections in 
  this projection are segments of a circle.

- The blade sections in the cylinder of figure 4.18-a are rotated into a plane parallel to 
  the propeller plane. The angle of rotation at each radius is the pitch angle at that 
  radius. This angle may vary over the radius; so it is a developed view. The ends 
  of the developed blade sections form the **developed blade contour**. The blade 
  sections in this projection remain circular.

- The circular blade sections in the developed contour can be expanded into a plane. 
  The contour thus derived is the developed and expanded contour, generally indicated 
  only as the **expanded contour**. The chord line of the sections in this contour is 
  now straight. Its area is called the **expanded blade area**.

### Cavitation

Cavitation inception occurs when the local pressure is equal to the vapor pressure. The local 
pressure is expressed in non-dimensional terms as the pressure coefficient $C_p$. Similarly, 
the **cavitation number** is expressed non-dimensionally as:

$$\sigma = \frac{p - p_v}{\frac{1}{2} \rho V^2}$$  \hspace{1cm} (4.61)

where $p$ is the undisturbed pressure in the flow at the propeller shaft $p_0 + \rho gh_s$, $p_v$ is the 
vapor pressure of water, $\rho$ is the density of water and $V$ is the undisturbed velocity in the 
fluid.

The condition that cavitation occurs when the local pressure is equal to the vapor pressure, 
means that a profile or a propeller will start to cavitate when the lowest pressure is near 
the vapor pressure. This is expressed in non-dimensional terms as:

$$\sigma = -C_p \text{(min)}$$  \hspace{1cm} (4.62)

The cavitation number or cavitation index, $\sigma$, is non-dimensional. This means that it is 
the parameter which has to be maintained when propeller model tests are carried out. It 
determines the pressure in the test section of the cavitation tunnel. 
Preservation of the cavitation number in a physical "ship+propeller" model is difficult. It 
is largely for this reason that MARIN built a towing tank in a vacuum chamber, about 
two decades ago. This allowed them to reduce the atmospheric pressure and thus to keep 
the $\sigma$-value in their model more like that in the prototype.
4.10. THRUST AND PROPULSION

Blade Area Ratio

An important parameter of the propeller is the **blade area ratio**, given as the ratio between the area of all blades and the area of the propeller plane, $A_0 = 0.25\pi D^2$:
- the projected blade area ratio, $A_P/A_0$ and
- the expanded blade area ratio, $A_E/A_0$

in which $A_P$ is the projected blade area and $A_E$ is the expanded blade area.

The latter, the **expanded blade area ratio**, $A_E/A_0$, is physically most significant; when no further indication is given, this blade area ratio is generally meant.

From the frictional resistance point of view, the blade area should be as low as possible. But a low blade area will result in low pressures (to obtain the required thrust) with high risks on cavitation. The blade area has to be chosen such that cavitation is avoided as much as possible.

Figure 4.20 shows the pressure distribution on a cavitating foil. The pressure at the nose on the back of the propeller drops down to the vapor pressure; a local cavity is the result. A lower pitch ratio (resulting in a less low pressure at the back) and a higher blade area ratio (so a larger foil area) will deliver the same thrust (with a bit lower efficiency) without cavitation.

![Figure 4.20: Cavitating Foil](image)

An old and very simple formula for the minimum **projected blade area ratio** is that by Taylor:

$$\frac{A_P}{A_0} = 1.067 - 0.229 \cdot \frac{P}{D} \quad (4.63)$$

A handsome indication for the minimum **expanded blade area ratio** gives the formula of Auf ’m Keller:

$$\frac{A_E}{A_0} = \frac{(1.3 + 0.3 \cdot Z) \cdot T}{(p - p_v) \cdot D^2} + k \quad (4.64)$$

in which the constant $k$ varies from 0.00 for fast naval vessels to 0.20 for highly powered full ships.
Number of Propeller Blades
The number of propeller blades, \( Z \), is chosen in relation to possible vibrations. An 8 cylinder engine and a 4 bladed propeller may suffer from resonance frequencies because the blade passage frequency (the frequency at which a blade passes close to the rudder for example) and the engine frequencies have common harmonics. In that case vibrations can become extensive, resulting in damage.
The structure of the wake is also important for the choice of the number of blades. When the wake has strong harmonics equal to the number of blades, thrust variations may become large; these, too, can lead to vibrations.

Wageningen Propeller Series
Systematic series of propeller models have been tested by MARIN to form a basis for propeller design. The starting point of a series is its parent form. The extent and applicability of the series depends on the parameters which are varied and on the range of their variations. Several series are described in the literature.
The most extensive and widely used fixed pitch open propeller series is the Wageningen B-series. The basic form of the B-series is simple and it has a good performance. The extent of the series is large; some 210 propellers have been tested.
The following parameters of the B-series propellers have been varied:
- the expanded blade area ratio, \( A_E/A_0 \), from 0.3 to 1.2
- the number of blades, \( Z \), from 2 to 7
- the pitch ratio, \( P/D \), from 0.5 to 1.4
The propellers are indicated by their number of blades and expanded blade area ratio. For instance, propeller B-4.85 has four blades and an expanded blade area ratio of 0.85. The test results are given in so-called open water diagrams per series of one blade number, one area ratio and a range of pitch ratios. The open water tests of the B-series were done at various rotation rates, \( n \), so at various Reynolds numbers. These results were corrected to a standard Reynolds number of \( Rn = 2 \cdot 10^6 \) along the lines of the ITTC method, as given by equation 4.42. However, these corrections are very small.
In addition to open fixed pitch propellers, MARIN has also developed series of fixed pitch propellers in an accelerating nozzle (\( K_n \)-series) and series of controllable pitch propellers in a nozzle. All available information on MARIN systematic propeller series has been summarized by [Kuiper, 1992]. The propulsive characteristics have been described by polynomials to make them usable in computer programs. Some other research institutes have also made their own propeller series, but the Wageningen Propeller Series are the most well known.

4.10.3 Propeller Mechanics
This section discusses the mechanics of a propeller in a uniform flow - the so-called free running or open water condition.

Open Water Characteristics
Suppose an open water propeller translating with a constant forward speed, \( V_e \), and a constant number of revolutions per second, \( n \), is immersed in a homogeneous initially stationary fluid without any currents or waves. Then, two velocity components can be
4.10. THRUST AND PROPULSION

defined for this propeller: a rotational velocity, $2\pi nr$, at a certain radius, $r$, and an axial translation velocity, $V_e$; see also figure 4.19-b. The hydrodynamic pitch angle, $\beta$, of the flow relative to the blade sections is generally taken at $r = 0.7R$, just as was done to define the pitch:

$$\beta_{0.7R} = \arctan\left(\frac{V_e}{0.7\pi \cdot nD}\right)$$  (4.65)

An important parameter to describe the angle of attack, and therefore the lift and drag, is the advance ratio, $J$, defined by:

$$J = \frac{V_e}{nD}$$  (4.66)

The resistance of a ship was made non-dimensional in equation 4.43 by:

$$C_t = \frac{R}{\frac{1}{2} \rho V^2 S} = \frac{R}{\frac{1}{4} \rho \cdot (\text{velocity})^2 \cdot (\text{area})}$$

When using the rotational velocity at for instance $0.7R$ as a characteristic velocity and the area of the propeller disk as a characteristic area, the thrust of a propeller can be made non-dimensional in the same way by:

$$C_T = \frac{T}{\frac{1}{2} \rho \cdot (0.7\pi \cdot nD)^2 \cdot \left(\frac{\pi D}{4}\right)^2 \cdot \rho D^4 n^2} \approx \frac{16.33}{\pi^3} \cdot \frac{T}{\rho D^4 n^2}$$  (4.67)

The constant $16.33/\pi^3$ can be included in the constant $C_T$ and so the thrust coefficient becomes:

$$K_T = \frac{T}{\rho D^4 n^2} \quad \text{and thrust becomes:} \quad T = K_T \cdot \rho D^4 n^2$$  (4.68)

The torque coefficient can be written in a similar way as:

$$K_Q = \frac{Q}{\rho D^5 n^2} \quad \text{and torque becomes:} \quad Q = K_Q \cdot \rho D^5 n^2$$  (4.69)

in which:

$$K_T = \text{thrust coefficient (-)}$$
$$K_Q = \text{torque coefficient (-)}$$
$$T = \text{thrust (N)}$$
$$Q = \text{torque (Nm)}$$
$$\rho = \text{density of water (kg/m}^3)$$
$$D = \text{diameter (m)}$$
$$n = \text{revolution speed (1/s)}$$

These propeller performance characteristics, $K_T$ and $K_Q$, in a uniform flow are given in figure 4.21.

The power delivered to the propeller is the delivered power $P_D$:

$$P_D = Q \cdot 2\pi n$$  (4.70)
CHAPTER 4. CONSTANT REAL FLOW PHENOMENA

Figure 4.21: Open Water Propeller Diagram

The power delivered by the thrust is the effective power $P_E$:

$$P_E = T \cdot V_e$$  \hspace{1cm} (4.71)

The efficiency of the open water propeller is the ratio between effective and delivered power:

$$\eta_O = \frac{P_E}{P_D} = \frac{T \cdot V_e}{Q \cdot 2\pi n}$$  \hspace{1cm} (4.72)

or:

$$\eta_O = \frac{K_T}{K_Q} \frac{J}{2\pi}$$  \hspace{1cm} (4.73)

In addition to the thrust and torque coefficients, $K_T$ and $K_Q$, the propulsive efficiency of the open water propeller, $\eta_O$, is shown in figure 4.21 too.

Four Quadrant Measurements

The $K_T$-$K_Q$-$J$ curves of open water propellers are restricted to the forward advance operation - that is the condition of forward speed of the ship and forward thrust of the propeller. During stopping and maneuvering of ships and for thrusters used for dynamic positioning, it is important to know the characteristics of the propellers under other conditions. The "four quadrants" of a series of B-propellers have been measured for that purpose. An example is given in figure 4.22-a.

The thrust and torque coefficients, $C_T$ and $C_Q$, in figure 4.22-b have been made non-dimensional axis using the undisturbed incoming velocity of the blade sections, $V_p$. Each curve is for a propeller with a different pitch ratio. The hydrodynamic pitch angle, $\beta$, is along the horizontal axis. Both the pitch ratio and $\beta$ have been determined at $r = 0.7R$. 
4.10. THRUST AND PROPULSION

Figure 4.22: Quadrant Measurement Results of B4-70 Propellers

\[ \beta = \arctan \left( \frac{V_e}{0.7\pi \cdot nD} \right) \]
\[ V_r = \sqrt{V_e^2 + (0.7\pi nD)^2} \]
\[ C_T^* = \frac{T}{\frac{1}{2} \rho \cdot V_r^2 \cdot \frac{\pi}{4} D^2} \quad \text{or} \quad T = C_T^* \cdot \frac{1}{2} \rho \cdot V_r^2 \cdot \frac{\pi}{4} D^2 \]
\[ C_Q^* = \frac{Q}{\frac{1}{2} \rho \cdot V_r^2 \cdot \frac{\pi}{4} D^3} \quad \text{or} \quad Q = C_Q^* \cdot \frac{1}{2} \rho \cdot V_r^2 \cdot \frac{\pi}{4} D^3 \quad (4.74) \]

The value of \( \beta \) varies over 360 degrees. From 0 until approximately 40 degrees, the curves are the regular open water diagrams as treated in the previous section. The condition with \( V_e = 0, \beta = 0 \) and \( n > 0 \) is the so-called bollard condition. The different combinations of directions of \( V_e \) and \( n \), so the four quadrants of \( \beta \), are listed below:
4.10.4 Ship Propulsion

This section treats the behavior of the propeller behind the ship and its interaction with the ship.

Wake Fraction

The velocity deficit behind the ship is a measure of its still water resistance. The velocity deficit in the propeller plane (without the propeller present) can be integrated and averaged over the propeller plane. This velocity is the average entrance velocity, \( V_e \), in the propeller plane when the propeller is absent. It is defined in terms of the ship speed, \( V_s \), by the nominal wake fraction, \( w_n \):

\[
w_n = \frac{V_s - V_e}{V_s}
\]

(4.75)

This definition is a non-dimensional form of the velocity deficit, \( V_s - V_e \), in the propeller plane.

The wake distribution is responsible for unsteadiness in the propeller loading during one revolution, as can be seen in figure 4.23. This figure shows the lines with a constant wake fraction - thus lines with constant axial velocities of the fluid particles - at the propeller disc behind various stern types of ships.

Unsteady effects will be neglected now and the (average) nominal wake fraction will be used to obtain the (constant) open water propeller inflow, which yields the entrance velocity:

\[
|V_e| = |V_s| \cdot (1 - w) \quad \text{with: } w = w_n
\]

(4.76)

For a statistical prediction of the wake fraction of ships, reference is given to regression formulas of [Holtrop, 1984].

Thrust Deduction Fraction

The propeller has an effect on the ship’s resistance, however. It increases the resistance of the ship by increasing the velocity along the hull (generally a small effect) and by decreasing the pressure around the stern. The thrust to be developed by the propeller should thus be greater than the resistance without propeller at the design speed, because the thrust has to be equal to the increased resistance.

The increase of resistance due to the propeller action is expressed as the thrust deduction fraction, \( t \):
4.10. THRUST AND PROPULSION

Figure 4.23: Wakes behind Various Stern Shapes

\[ t = \frac{T - R}{T} \quad (4.77) \]

where \( T \) is the thrust needed to maintain a certain design speed and \( R \) is the resistance without propeller at that speed, as found from resistance tests.

With this, the relation between resistance and thrust is:

\[ R = T \cdot (1 - t) \quad (4.78) \]

For a statistical prediction of the thrust deduction fraction of ships, reference is given to regression formulas of [Holtrop, 1984].

**Propulsive Efficiency**

The total efficiency is the ratio of the useful energy delivered by a system (output) and the energy supplied to the system (input). For the ship the output is \( R \cdot V_s \) and the input is \( Q \cdot \omega = Q \cdot 2\pi n \).

So the **total propulsive efficiency** is:

\[ \eta_T = \frac{R \cdot V_s}{Q \cdot 2\pi n} \quad (4.79) \]

This total efficiency can be divided into parts which are related to the propeller performance without the hull and to the hull without the propeller:
\[ \eta_T = \frac{R \cdot V_s}{Q \cdot 2\pi n} = \frac{T(1 - t) \cdot \frac{V_s}{1 - w}}{Q_O \cdot Q_O \cdot 2\pi n} = \left( \frac{T \cdot V_e}{Q_O \cdot 2\pi n} \right) \cdot \left( \frac{1 - t}{1 - w} \right) \cdot \left( \frac{Q_O}{Q} \right) = \left( \frac{K_T \cdot J}{K_Q \cdot 2\pi} \right) \cdot \left( \frac{1 - t}{1 - w} \right) \cdot \left( \frac{Q_O}{Q} \right) \]

(4.80)

or:

\[ |\eta_T = \eta_O \cdot \eta_H \cdot \eta_R | \]

in which, at the same thrust, \( Q_O \) is the torque of the open water propeller in a uniform flow and \( Q \) is the torque of the propeller in the wake behind the ship.

The total propulsive efficiency is thus divided into three components:

- **Open Water Efficiency:**

\[ \eta_O = \frac{T \cdot V_e}{Q_O \cdot 2\pi n} = \frac{K_T \cdot J}{K_Q \cdot 2\pi} \]

(4.81)

This is the efficiency of the propeller alone in the mean (homogeneous) inflow, \( V_e \). It can be derived from open water diagrams of propellers.

- **Hull Efficiency:**

\[ \eta_H = \frac{R \cdot V_s}{T \cdot V_e} = \frac{1 - t}{1 - w} \]

(4.82)

Old but convenient rough approximations of the wake fraction and the thrust deduction fraction of full scale ships are given by:

\[ w \approx 0.5 \cdot C_B - 0.05 \]
\[ t \approx 0.6 \cdot w \]

(4.83)

where \( C_B \) is the block coefficient of the ship. A fast slender container vessel with \( C_B = 0.55 \) will have \( \eta_H \approx 1.12 \) while for a crude oil carrier with \( C_B = 0.85, \eta_H \approx 1.24 \). So the effect of a hull with its wake before the propeller increases the propulsive efficiency considerably. The propeller diameter therefore has to be such that the wake is going through the propeller disk as much as possible.

When using model data, it should be noted that - contrarily to the thrust deduction fraction - the wake fraction is very sensitive for scale effect. It is left to the reader to explain this.

- **Relative Rotative Efficiency:**

\[ \eta_R = \frac{Q_O}{Q} \]

(4.84)
This efficiency reflects the difference in torque in the wake and in open water at the same thrust. The relative rotative efficiency is generally close to one; $\eta_R = 0.98 - 1.00$ for single-screw ships and $\eta_R = 1.00 - 1.02$ for twin-screw ships. See [Holtrop, 1984] for a statistical prediction of the relative rotative efficiency.

4.10.5 Propulsion versus Resistance

After completing the vessel, full scale trial tests will be carried out to verify the speed of the ship. An example of trial results is given in figure 4.24. This figure shows a comparison of predicted (from model tank test results) and measured (from full scale trial tests) data on the revolutions per minute ($RPM$) of the propeller and the power delivered to the propeller ($SHP = \text{Shaft Horse Power}$) for different ship speeds of a pilot vessel in calm water.

![Figure 4.24: Trial Results of a Pilot Vessel.](image)

A very practical offshore engineering application of the information from the latter sections of this chapter involves the prediction of the speed at which a barge (or other floating object) will be towed by a given tugboat. Such information can be invaluable for the logistic planning of a major offshore operation; consider the costs of keeping a larger semi-submersible crane vessel waiting (at perhaps a quarter million dollars per day) for a barge - with topsides to install - which arrives a bit later than planned.

A tugboat will of course be able to deliver more thrust than it needs to overcome its own frictional and wave making resistance. In general, the available **towing force** which a tugboat can deliver will be a function of its towing speed. This function, which decreases with increasing speed, will be known for the tug selected. In general, each tug will have a family of curves depending upon the speed of its engine. On the other hand, only one engine speed will deliver the highest overall efficiency for any given speed.

The resistance for the towed object should be known as well. This resistance force will generally be an increasing function of towing velocity. Superposition of the two curves -
one for the tugboat and one for the towed object will yield the optimum towing speed. This is the speed corresponding to the intersection point of the two curves; see figure 4.25.

Figure 4.25: Free Running and Towing Speed of a Tug

### 4.10.6 Lift and Flettner Rotors

As early as the eighteenth century, artillery officers noted that spinning shells fired in a cross-wind often either overshot or undershot their targets. In 1794, the Berlin Academy offered a prize for the explanation of this. It was more than 50 years before Gustav Magnus, a Professor of Physics in Berlin conducted experiments with spinning cylinders in air to demonstrate the phenomenon.

Little was done with this information until 1922 when Anton Flettner, a German businessman, set out to apply the Magnus effect (in air) for the propulsion of a ship. Since then, Flettner rotors have been used experimentally for steering ships; they are becoming popular for steering towed remote controlled vehicles.

Flettner rotors are circular cylinders which spin in an air or water flow. Because there is friction between their surface and the surrounding fluid, they cause a circulation and thus a lift force. Their operating principle based upon the Magnus effect was given in chapter 3; this section concentrates on their more practical aspects.

It was shown at the end of chapter 3 that the lift force generated by a unit length of Flettner rotor with a radius $R$ is:

$$F_L = 4\pi R \rho V^2 C$$

(4.85)

in which:

- $F_L$ = lift force per unit length (N/m)
- $V$ = towing velocity (m/s)
- $C$ = dimensionless circulation strength (-)
- $R$ = rotor radius (m)

$C$ relates the circulation strength, $\Gamma$, to $R$ and $V$ via:
4.10. THRUST AND PROPULSION

Figure 4.26: Sailboat Prototype based on Flettner Rotor Principle

\[ C = -\frac{\Gamma}{4\pi RV} \]  
(4.86)

as given in chapter 3.

It is logical to assume that \( \Gamma \) and thus \( C \) will be directly related to the speed of the Flettner cylinder surface, \( R \omega \), in which \( \omega \) is the angular velocity of the rotor in radians per second. When all this is substituted into the equation for \( F_L \), one finds that the total lift force generated by a rotor will be proportional to four independent quantities which the designer or user has at his or her discretion:

\[ F_L \propto R \omega V \ell \]  
(4.87)

in which:

- \( \ell \) = length of the rotor (m)
- \( R \) = rotor radius (m)
- \( V \) = towing velocity (m/s)
- \( \omega \) = angular velocity (rad/s)

The above equation is stated as a proportionality rather than an equality; it does not include the water density nor does it include coefficients to include the fact that the flow is not ideal. Indeed, no hard quantitative data has been given here. Questions such as:
- How does the spin of the rotor influence its drag?
- Can a Flettner rotor also have a cavitation problem?
- What is the relation between spin speed and torque on the one hand and the delivered circulation on the other hand?

can be asked.

The answers to these questions (among others) will depend upon experimental evidence which does not yet seem to be available in the open literature just yet. Indeed, this application of Flettner rotors is still quite new!
On a small scale, Flettner rotors are being used in the offshore industry to steer RCV’s. On a larger scale, sailboats without sails have even been proposed! This design has large diameter vertical cylinders mounted vertically from the deck instead of masts. They are spun about their vertical axes by motors thus setting up a lift force in the wind much like a spinning tennis ball would. This method of propulsion has not yet proven practical at this scale, however, although prototypes have been built as shown in figure 4.26.
Chapter 5

OCEAN SURFACE WAVES

5.1 Introduction

Ocean surface waves cause periodic loads on all sorts of man-made structures in the sea. It does not matter whether these structures are fixed or floating and on the surface or deeper in the sea.

Most structures, even the so-called fixed structures, are not really rigid; they respond in some way to the wave-induced periodic loads. Examples of such response include accelerations, harmonic displacements and internal loads in ships as well as fixed structures. Waves and the resulting ship motions cause added resistance, reduced sustained speed (with associated longer travel times) and increased fuel consumption for ships. Also local erosion near pipelines and other small structures on the sea bed can be caused by waves. The ship’s responses can impair safety via phenomena such as shipping water on deck or wave slamming.

In short, there is every reason to need to know about waves - the topic of this chapter - before one goes on with the discussion of all sorts of hydrodynamic problems in later chapters.

Water waves can be generated in many different ways:

- Waves generated by a ship or any other floating structure which is moving, either at a constant forward speed or by carrying out an oscillatory motion.
- Waves generated by the interaction between wind and the sea surface.
- Waves generated by astronomical forces: Tides.
- Waves generated by earthquakes or submarine landslides: Tsunamis.
- Free surface waves generated in fluids in partially filled tanks; such as fuel or cargo tanks on a ship.

A single mathematical solution for all problems related to these various types of waves does not exist; even in simple cases approximations are required. It is important to be aware of the limitations of simplifications, especially when nonlinear effects can become important. On the other hand, a simple linear approach can work quite well for many practical applications.

Wind generated waves can be classified into two basic categories: sea and swell. These are described briefly here.

---

- **Sea**
  A sea is a train of waves driven by the prevailing local wind field. The waves are short-crested with the lengths of the crests only a few (2-3) times the apparent wave length. Also, sea waves are very irregular; high waves are followed unpredictably by low waves and vice versa. Individual wave crests seem to propagate in different directions with tens of degrees deviation from the mean direction. The crests are fairly sharp and sometimes even small waves can be observed on these crests or there are dents in the larger wave crests or troughs. The apparent or virtual wave period, $T$, varies continuously, as well as the virtual or apparent wave length, $\lambda$.

- **Swell**
  A swell is waves which have propagated out of the area and local wind in which they were generated. They are no longer dependent upon the wind and can even propagate for hundreds of kilometers through areas where the winds are calm. Individual waves are more regular and the crests are more rounded than those of a sea. The lengths of the crests are longer, now several (6-7) times the virtual wave length. The wave height is more predictable, too. If the swell is high, 5 to 6 waves of approximately equal heights can pass a given point consecutively. If the waves are low, they can stay low for more than a minute even though the surface elevation remains irregular.

Wind waves, especially, are very irregular. Even so, they can be seen as a superposition of many simple, regular harmonic wave components, each with its own amplitude, length, period or frequency and direction of propagation. Such a concept can be very handy in many applications; it allows one to predict very complex irregular behavior in terms of much simpler theory of regular waves. This so-called **superposition principle**, first introduced in hydrodynamics by [St. Denis and Pierson, 1953], is illustrated in figure 5.1. To analyze complicated wave systems, it is necessary to know the properties of the simple harmonic components, such as time and location-dependent pressure in the fluid, relation between wave length and wave period, energy transport, etc.

In this respect waves are often classified into two other basic categories:

- **Deep water waves**, also sometimes referred to as **short waves**
  The water is considered to be deep if the water depth, $h$, is more than half the wave length, $\lambda$, so: $h/\lambda > 1/2$ or $\lambda/h < 2$. These (relatively) short waves do not ‘feel’ the sea floor.

- **Shallow water waves**, also sometimes referred to as **long waves**
  The water is considered to be shallow if the water depth, $h$, is less than $1/20$ of the wave length, $\lambda$, so: $h/\lambda < 1/20$ or $\lambda/h > 20$. The sea floor has a very large influence on the characteristics of these (relatively) long waves.

### 5.2 Regular Waves

Definitions used to describe the harmonic - or regular - waves and the potential theory used to solve the flow problem in waves will be treated now.

Figure 5.2 shows a harmonic wave as seen from two different perspectives. Figure 5.2-a shows what one would observe in a snapshot photo made looking at the side of a (transparent) wave flume; the wave profile is shown as a function of distance $x$ along the flume at a fixed instant in time. Figure 5.2-b is a time record of the water level observed at one
location along the flume; it looks similar in many ways to the other figure, but time $t$ has replaced $x$ on the horizontal axis.

Notice that the origin of the coordinate system is at the still water level with the positive $z$-axis directed upward; most relevant values of $z$ will be negative. The still water level is the average water level or the level of the water if no waves were present. The $x$-axis is positive in the direction of wave propagation. The water depth, $h$, (a positive value) is measured between the sea bed ($z = -h$) and the still water level.

The highest point of the wave is called its crest and the lowest point on its surface is the trough. If the wave is described by a sine wave, then its amplitude $\zeta_a$ is the distance from the still water level to the crest, or to the trough for that matter. The subscript $a$ denotes amplitude, here. The wave height $H$ is measured vertically from wave trough level to the wave crest level. Obviously:

$$|H = 2\zeta_a|$$  

for a sinusoidal wave \hspace{1cm} (5.1)

The horizontal distance (measured in the direction of wave propagation) between any two successive wave crests is the wave length, $\lambda$. The distance along the time axis is the wave period, $T$. The ratio of wave height to wave length is often referred to as the dimensionless
CHAPTER 5. OCEAN SURFACE WAVES

Figure 5.2: Harmonic Wave Definitions

wave steepness, \( H/\lambda \).
Since the distance between any two corresponding points on successive sine waves is the same, wave lengths and periods are usually actually measured between two consecutive upward (or downward) crossings of the still water level. Such points are also called zero-crossings, and are easier to detect in a wave record.
Since sine or cosine waves are expressed in terms of angular arguments, the wave length and period are converted to angles using:

\[
k = \frac{2\pi}{\lambda} \quad \text{or} \quad \lambda = \frac{2\pi}{k}
\]

\[
\omega = \frac{2\pi}{T} \quad \text{or} \quad T = \frac{2\pi}{\omega}
\]

in which \( k \) is the wave number (rad/m) and \( \omega \) is the circular wave frequency (rad/s).
Obviously, the wave form moves one wave length during one period so that its speed or phase velocity, \( c \), is given by:

\[
c = \frac{\lambda}{T} = \frac{\omega}{k}
\]

If the wave moves in the positive \( x \)-direction, the wave profile - the form of the water surface - can now be expressed as a function of both \( x \) and \( t \) as follows:

\[
\zeta = \zeta_a \cos(kx - \omega t)
\]

A wave moving in the opposite (negative \( x \)-direction can be given by:

\[
\zeta = \zeta_a \cos(kx + \omega t)
\]

A progressive harmonic wave is shown in figure 5.3.

5.2.1 Potential Theory
The basics of potential flow theory used here are no different from those presented in chapter 3. In order to use this linear theory with waves, it will be necessary to assume that
5.2. REGULAR WAVES

Figure 5.3: Progressive Harmonic Wave

the water surface slope is very small. This means that the wave steepness is so small that terms in the equations of the waves with a magnitude in the order of the steepness-squared can be ignored. Using the linear theory holds here that harmonic displacements, velocities and accelerations of the water particles and also the harmonic pressures will have a linear relation with the wave surface elevation.

The profile of a simple wave with a small steepness looks like a sine or a cosine and the motion of a water particle in a wave depends on the distance below the still water level. This is reason why the wave potential is written as:

$$\Phi_w(x, z, t) = P(z) \cdot \sin(kx - \omega t)$$

(5.7)

in which $P(z)$ is an (as yet) unknown function of $z$.

The velocity potential $\Phi_v(x, z, t)$ of the harmonic waves has to fulfill four requirements:

1. Continuity condition or Laplace equation
2. Sea bed boundary condition
3. Free surface dynamic boundary condition
4. Free surface kinematic boundary condition.

These requirements lead to a more complete expression for the velocity potential as will be explained in the following sections. The relationships presented in these sections are valid for all water depths, but the fact that they contain so many hyperbolic functions makes them cumbersome to use. Engineers - as opposed to (some) scientists - often look for ways to simplify the theory. The simplifications stem from the following approximations for large and very small arguments, $s$, as shown in figure 5.4:

for large arguments, $s$

$$\sinh(s) \approx \cosh(s) \gg s$$
$$\tanh(s) \approx 1$$

(5.8)

for small arguments, $s$

$$\sinh(s) \approx \tanh(s) \approx s$$
$$\cosh(s) \approx 1$$

(5.9)

In memory:

$$\sinh s = \frac{e^s - e^{-s}}{2} \quad \cosh s = \frac{e^s + e^{-s}}{2} \quad \tanh s = \frac{e^s - e^{-s}}{e^s + e^{-s}}$$

(5.10)
CHAPTER 5. OCEAN SURFACE WAVES

Figure 5.4: Hyperbolic Functions Limits

Continuity Condition and Laplace Equation

The velocity of the water particles \((u, v, w)\) in the three translational directions, or alternatively \((v_x, v_y, v_z)\), follow from the definition of the velocity potential, \(\Phi_w\):

\[
\begin{align*}
    u &= v_x = \frac{\partial \Phi_w}{\partial x} \\
    v &= v_y = \frac{\partial \Phi_w}{\partial y} \\
    w &= v_z = \frac{\partial \Phi_w}{\partial z}
\end{align*}
\]  
(5.11)

The two-dimensional version of this can be found in chapter 3.

Since the fluid is homogeneous and incompressible, the Continuity Condition:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

results in the Laplace Equation for potential flows:

\[
\nabla^2 \Phi_w = \frac{\partial^2 \Phi_w}{\partial x^2} + \frac{\partial^2 \Phi_w}{\partial y^2} + \frac{\partial^2 \Phi_w}{\partial z^2} = 0
\]

(5.13)

Water particles move here in the \(x-z\) plane only, so in the equations above:

\[
\begin{align*}
    v &= \frac{\partial \Phi_w}{\partial y} = 0 \\
    \frac{\partial v}{\partial y} &= \frac{\partial^2 \Phi_w}{\partial y^2} = 0
\end{align*}
\]

(5.14)

Taking this into account, a substitution of equation 5.7 in equation 5.13 yields a homogeneous solution of this equation:

\[
\frac{d^2 P(z)}{dz^2} - k^2 P(z) = 0
\]

(5.15)

with as solution for \(P(z)\):

\[
P(z) = C_1 e^{+kz} + C_2 e^{-kz}
\]

(5.16)

Using this result from the first boundary condition, the wave potential can be written now with two unknown coefficients as:

\[
\Phi_w(x, z, t) = (C_1 e^{+kz} + C_2 e^{-kz}) \cdot \sin (kx - \omega t)
\]

(5.17)

in which:
\[ \Phi_w(x, z, t) = \text{wave potential (m}^2/\text{s}) \]
\[ e = \text{base of natural logarithms (-)} \]
\[ C_1, C_2 = \text{as yet undetermined constants (m}^2/\text{s}) \]
\[ k = \text{wave number (1/m)} \]
\[ t = \text{time (s)} \]
\[ x = \text{horizontal distance (m)} \]
\[ z = \text{vertical distance, positive upwards (m)} \]
\[ \omega = \text{wave frequency (1/s)} \]

**Sea Bed Boundary Condition**

The vertical velocity of water particles at the sea bed is zero (no-leak condition):

\[ \frac{\partial \Phi_w}{\partial z} = 0 \quad \text{for: } z = -h \quad (5.18) \]

as given in figure 5.5.

![Figure 5.5: Vertical Velocity at the Sea Bed](image)

Substituting this boundary condition in equation 5.17 provides:

\[ kC_1 e^{-kh} - kC_2 e^{+kh} = 0 \quad (5.19) \]

or:

\[ C_1 e^{-kh} = C_2 e^{+kh} \quad (5.20) \]

By defining:

\[ C = C_2 e^{+kh} \quad (5.21) \]

or:

\[ C_1 = \frac{C}{2} e^{+kh} \quad \text{and} \quad C_2 = \frac{C}{2} e^{-kh} \quad (5.22) \]

it follows that \( P(z) \) in equation 5.16 can be worked out to:

\[ P(z) = \frac{C}{2} \left( e^{+k(h+z)} + e^{-k(h+z)} \right) = \frac{C}{2} \text{cosh} \, k \, (h + z) \quad (5.23) \]

and the wave potential with only one unknown becomes:

\[ \Phi_w(x, z, t) = C \cdot \text{cosh} \, k \, (h + z) \cdot \sin \, (kx - \omega t) \quad (5.24) \]

in which \( C \) is an (as yet) unknown constant.
Free Surface Dynamic Boundary Condition

The pressure, $p$, at the free surface of the fluid, $z = \zeta$, is equal to the atmospheric pressure, $p_0$. This requirement for the pressure is called the dynamic boundary condition at the free surface, see figure 5.6.

![Figure 5.6: Atmospheric Pressure at the Free Surface](image)

The Bernoulli equation for an unstationary irrotational flow (with the velocity given in terms of its three components) is in its general form:

$$\frac{\partial \Phi_w}{\partial t} + \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} + gz = C^*$$

(5.25)

In two dimensions, $v = 0$ and since the waves have a small steepness ($u$ and $w$ are small), this equation becomes:

$$\frac{\partial \Phi_w}{\partial t} + \frac{p}{\rho} + gz = C^*$$

(5.26)

At the free surface this condition becomes:

$$\frac{\partial \Phi_w}{\partial t} + \frac{p_0}{\rho} + g\zeta = C^* \quad \text{for: } z = \zeta$$

(5.27)

The constant value $p_0/\rho - C^*$ can be included in $\partial \Phi_w/\partial t$; this will not influence the velocities being obtained from the potential $\Phi_w$. With this the equation becomes:

$$\frac{\partial \Phi_w}{\partial t} + g\zeta = 0 \quad \text{for: } z = \zeta$$

(5.28)

The potential at the free surface can be expanded in a Taylor series, keeping in mind that the vertical displacement $\zeta$ is relatively small:

$$\{\Phi_w(x, z, t)\}_{z=\zeta} = \{\Phi_w(x, z, t)\}_{z=0} + \zeta \cdot \left\{ \frac{\partial \Phi_w(x, z, t)}{\partial z} \right\}_{z=0} + \ldots$$

$$\left\{ \frac{\partial \Phi_w(x, z, t)}{\partial t} \right\}_{z=\zeta} = \left\{ \frac{\partial \Phi_w(x, z, t)}{\partial t} \right\}_{z=0} + O(\epsilon^2)$$

(5.29)

which yields for the linearized form of the free surface dynamic boundary condition:
5.2. REGULAR WAVES

\[ \frac{\partial \Phi_w}{\partial t} + g \zeta = 0 \quad \text{for: } z = 0 \]  

(5.30)

With this, the wave profile becomes:

\[ \zeta = -\frac{1}{g} \cdot \frac{\partial \Phi_w}{\partial t} \quad \text{for: } z = 0 \]  

(5.31)

A substitution of equation 5.24 in equation 5.31 yields the wave profile:

\[ \zeta = \frac{\omega C}{g} \cdot \cosh kh \cdot \cos (kx - \omega t) \]  

(5.32)

or:

\[ \zeta = \zeta_a \cdot \cos (kx - \omega t) \quad \text{with: } \zeta_a = \frac{\omega C}{g} \cdot \cosh kh \]  

(5.33)

With this the corresponding wave potential, depending on the water depth \( h \), is given by the relation:

\[ \Phi_w = \frac{\zeta_a g}{\omega} \cdot \frac{\cosh kh (h + z)}{\cosh kh} \cdot \sin(kx - \omega t) \]  

(5.34)

In deep water with \( h \to \infty \) (short waves), the wave potential becomes:

\[ \Phi_w = \frac{\zeta_a g}{\omega} \cdot e^{kh} \cdot \sin(kx - \omega t) \quad \text{(deep water)} \]  

(5.35)

Free Surface Kinematic Boundary Condition

So far the relation between the wave period \( T \) and the wave length, \( \lambda \), is still unknown. The relation between \( T \) and \( \lambda \) (or equivalently \( \omega \) and \( k \)) follows from the boundary condition that the vertical velocity of a water particle at the free surface of the fluid is identical to the vertical velocity of that free surface itself (no-leak condition); this is a kinematic boundary condition.

Using the equation of the free surface 5.33 yields for the wave surface:

\[ \frac{dz}{dt} = \frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x} \cdot \frac{dx}{dt} \quad \text{for: } z = \zeta \]

\[ = \frac{\partial \zeta}{\partial t} + u \cdot \frac{d\zeta}{dx} \]  

(5.36)

The second term in this expression is a product of two values, which are both small because of the assumed small wave steepness. This product becomes even smaller (second order) and can be ignored, see figure 5.7.

This linearization provides the vertical velocity of the wave surface:

\[ \frac{dz}{dt} = \frac{\partial \zeta}{\partial t} \quad \text{for: } z = \zeta \]  

(5.37)

The vertical velocity of a water particle in the free surface is then:

\[ \frac{\partial \Phi_w}{\partial z} = \frac{\partial \zeta}{\partial t} \quad \text{for: } z = \zeta \]  

(5.38)
Analogous to equation 5.30 this condition is valid for $z = 0$ too, instead of for $z = \zeta$ only. A differentiation of the free surface dynamic boundary condition (equation 5.30) with respect to $t$ provides:

$$\frac{\partial^2 \Phi_w}{\partial t^2} + g \frac{\partial \zeta}{\partial t} = 0 \quad \text{for } z = 0 \quad (5.39)$$

or after re-arranging terms:

$$\frac{\partial \zeta}{\partial t} + \frac{1}{g} \cdot \frac{\partial^2 \Phi_w}{\partial t^2} = 0 \quad \text{for } z = 0 \quad (5.40)$$

Together with equation 5.37 this delivers the **free surface kinematic boundary condition** or the **Cauchy-Poisson condition**:

$$\frac{\partial z}{\partial t} + \frac{1}{g} \cdot \frac{\partial^2 \Phi_w}{\partial t^2} = 0 \quad \text{for: } z = 0 \quad (5.41)$$

**Dispersion Relationship**

The information is now available to establish the relationship between $\omega$ and $k$ (or equivalently $T$ and $\lambda$) referred to above. A substitution of the expression for the wave potential (equation 5.34) in equation 5.41 gives the **dispersion relation** for any arbitrary water depth $h$:

$$\omega^2 = k \cdot g \cdot \tanh kh \quad (5.42)$$

In many situations, $\omega$ or $T$ will be know; one must determine $k$ or $\lambda$. Since $k$ appears in a nonlinear way in 5.42, that equation will generally have to be solved iteratively. **In deep water** ($\tanh kh = 1$), equation 5.42 degenerates to a quite simple form which can be used without difficulty:

$$\omega^2 = k \cdot g \quad \text{(deep water)} \quad (5.43)$$

and the deep water relation between $T$ and $\lambda$ becomes:

$$T = \sqrt{\frac{2\pi}{g}} \cdot \sqrt{\lambda} \quad \text{or} \quad \lambda = \frac{g}{2\pi} \cdot T^2 \quad \text{(deep water)} \quad (5.44)$$
5.2. REGULAR WAVES

Substitution of \( g = 9.81 \, \text{m/s}^2 \) and \( \pi \) yields:

\[
T \approx 0.80 \cdot \sqrt{\lambda} \quad \text{or} \quad \lambda \approx 1.56 \cdot T^2 \quad \text{(deep water)} \quad (5.45)
\]

Note that this regular wave relation cannot be used to describe the relation between the average wave length and the average wave period of an irregular sea. However in a more regular swell, this relation can be used with an accuracy of about 10 to 15 per cent.

**In shallow water**, the dispersion relation is found by substituting \( \tanh kh = kh \) in equation 5.42; thus:

\[
\omega = k \cdot \sqrt{gh} \quad \text{(shallow water)} \quad (5.46)
\]

and the shallow water relation between \( T \) and \( \lambda \) becomes:

\[
T = \frac{\lambda}{\sqrt{gh}} \quad \text{or} \quad \lambda = T \cdot \sqrt{gh} \quad \text{(shallow water)} \quad (5.47)
\]

**Cauchy-Poisson Condition in Deep Water**

In deep water (short waves), the free surface kinematic boundary condition or Cauchy-Poisson condition is often given in another form in the literature. The space and time dependent wave potential, \( \Phi_w(x, z, t) \), is divided in a space-dependent part, \( \phi_w(x, z) \), and a time-dependent part, \( 1 \cdot \sin \omega t \), by defining:

\[
\Phi_w(x, z, t) = \phi_w(x, z) \cdot \sin \omega t \quad (5.48)
\]

Equation 5.41:

\[
\frac{\partial z}{\partial t} + \frac{1}{g} \cdot \frac{\partial^2 \Phi_w}{\partial t^2} = 0 \quad \text{for: } z = 0
\]

can with equation 5.37 be written as:

\[
\frac{\partial \zeta}{\partial t} + \frac{1}{g} \cdot \frac{\partial^2 \Phi_w}{\partial t^2} = 0 \quad \text{for: } z = 0 \quad (5.49)
\]

and using equation 5.38 results in:

\[
\frac{\partial \Phi_w}{\partial z} + \frac{1}{g} \cdot \frac{\partial^2 \Phi_w}{\partial t^2} = 0 \quad \text{for: } z = 0 \quad (5.50)
\]

With the dispersion relation in deep water, \( \omega^2 = k \cdot g \), the free surface kinematic boundary condition or the Cauchy-Poisson condition becomes:

\[
\frac{\partial \phi_w}{\partial z} - k \cdot \phi_w = 0 \quad \text{for } z = 0 \quad \text{(deep water)} \quad (5.51)
\]

**5.2.2 Phase Velocity**

With the dispersion relation (equation 5.42) the wave celerity \((c = \lambda/T = \omega/k)\) becomes:

\[
c = \sqrt{\frac{g}{k} \cdot \tanh kh} \quad (5.52)
\]
The phase velocity increases with the wave length \( k = 2\pi/\lambda \); water waves display dispersion in that longer waves move faster than shorter ones. As a result of this phenomena, sea sailors often interpret swell (long, relatively low wind-generated waves which have moved away from the storm that generated them) as a warning of an approaching storm. **In deep water,** the phase velocity is found by substituting \( \tanh kh = 1 \) in equation 5.52; thus:

\[
\frac{g}{k} \frac{g}{\omega} \quad \text{(deep water)} \quad (5.53)
\]

With some further substitution, one can get:

\[
\frac{g}{k} = \sqrt{\frac{g}{2\pi} \cdot \lambda} \approx 1.25\sqrt{\lambda} \approx 1.56 \cdot T \quad \text{(deep water)} \quad (5.54)
\]

**In shallow water,** the phase velocity is found by substituting \( \tanh kh = kh \) in equation 5.52; thus:

\[
\frac{g}{h} \quad \text{(shallow water)} \quad (5.55)
\]

The phase velocity is now independent of the wave period; these waves are not dispersive. This celerity, \( \sqrt{gh} \), is called the **critical velocity.** This velocity is of importance when sailing with a ship at shallow water. Similar effects occur with an airplane that exceeds the speed of sound. Generally, the forward ship speed will be limited to about 80% of the critical velocity, to avoid excessive still water resistance or **squat** (combined sinkage and trim). Indeed, only a ship which can plane on the water surface is able to move faster than the wave which it generates.

### 5.2.3 Water Particle Kinematics

The kinematics of a water particle is found from the velocity components in the \( x \)- and \( z \)-directions, obtained from the velocity potential given in equation 5.34 and the dispersion relation given in equation 5.42.

**Velocities**

The resulting velocity components - in their most general form - can be expressed as:

\[
u = \frac{\partial \Psi_w}{\partial x} = \frac{dx}{dt} = \zeta_a \cdot \frac{k g}{\omega} \cdot \frac{\cosh k (h + z)}{\cosh kh} \cdot \cos (kx - \omega t)\]

\[
w = \frac{\partial \Psi_w}{\partial z} = \frac{dz}{dt} = \zeta_a \cdot \frac{k g}{\omega} \cdot \frac{\sinh k (h + z)}{\cosh kh} \cdot \sin (kx - \omega t) \quad (5.56)
\]

A substitution of:

\[
k g = \frac{\omega^2}{\tanh kh}
\]

derived from the dispersion relation in equation 5.42, provides:
5.2. REGULAR WAVES

\[
\begin{align*}
    u &= \zeta_a \cdot \omega \cdot \frac{\cosh k(h+z)}{\sinh kh} \cdot \cos(kx-\omega t) \\
    w &= \zeta_a \cdot \omega \cdot \frac{\sinh k(h+z)}{\sin kh} \cdot \sin(kx-\omega t)
\end{align*}
\]

(5.57)

An example of a velocity field is given in figure 5.8.

Figure 5.8: Velocity Field in a Shallow Water Wave

**In deep water**, the water particle velocities are given by:

\[
\begin{align*}
    u &= \zeta_a \omega \cdot e^{kz} \cdot \cos(kx-\omega t) \\
    w &= \zeta_a \omega \cdot e^{kz} \cdot \sin(kx-\omega t)
\end{align*}
\]

(deep water) (5.58)

An example of a velocity field is given in figure 5.9.

Figure 5.9: Velocity Field in a Deep Water Wave

The circular outline velocity or **orbital velocity** in deep water (short waves) is:
CHAPTER 5. OCEAN SURFACE WAVES

\[ V_o = \sqrt{u^2 + w^2} = \zeta_a \omega \cdot e^{kz} \quad \text{(deep water)} \]  \hspace{1cm} (5.59)

**In shallow water**, the water velocity components are:

\[ u = \zeta_a \omega \cdot \frac{1}{kh} \cdot \cos (kx - \omega t) \]
\[ w = \zeta_a \omega \cdot (1 + \frac{z}{h}) \cdot \sin (kx - \omega t) \quad \text{(shallow water)} \]  \hspace{1cm} (5.60)

**Displacements**

Because of the small steepness of the wave, \( x \) and \( z \) in the right hand side of these equations can be replaced by the coordinates of the mean position of the considered water particle: \( x_1 \) and \( z_1 \). Hence the distances \( x - x_1 \) and \( z - z_1 \) are so small that differences in velocities resulting from the water motion position shifts can be neglected; they are of second order. Then, an integration of equations 5.57 over \( t \) yields the water displacements:

\[ x = -\zeta_a \cdot \frac{\cosh k(h + z_1)}{\sinh kh} \cdot \sin (kx_1 - \omega t) + C_1 \]
\[ z = +\zeta_a \cdot \frac{\sinh k(h + z_1)}{\sinh kh} \cdot \cos (kx_1 - \omega t) + C_2 \]  \hspace{1cm} (5.61)

**Trajectories**

It is obvious that the water particle carries out an oscillation in the \( x \)- and \( z \)-directions about a point \((C_1, C_2)\). This point will hardly deviate from the situation in rest, so: \( C_1 \approx x_1 \) and \( C_2 \approx z_1 \).

The trajectory of the water particle is found by an elimination of the time, \( t \), by using:

\[ \sin^2 (kx_1 - \omega t) + \cos^2 (kx_1 - \omega t) = 1 \]  \hspace{1cm} (5.62)

which provides:

\[ \frac{(x - x_1)^2}{(\zeta_a \cdot \frac{\cosh k(h + z_1)}{\sinh kh})^2} + \frac{(z - z_1)^2}{(\zeta_a \cdot \frac{\sinh k(h + z_1)}{\sinh kh})^2} = 1 \]  \hspace{1cm} (5.63)

This equation shows that the trajectories of water particles are ellipses in the general case, as shown in figure 5.10. The water motion obviously decreases as one moves deeper below the water surface.

Half the vertical axis of the ellipse - the vertical water displacement amplitude - is equal to \( \zeta_a \) at the free surface, \( z_1 = 0 \). This amplitude is reduced to zero at the sea bed, \( z_1 = -h \), in accordance with the sea bed boundary condition; the trajectories become horizontal line elements there.

**In deep water**, the trajectories of the water particles become circles:

\[ (x - x_1)^2 + (z - z_1)^2 = (\zeta_a \cdot e^{kz_1})^2 \quad \text{(deep water)} \]  \hspace{1cm} (5.64)
5.2. REGULAR WAVES

Figure 5.10: Trajectories of Water Particles in Long or Shallow Water Waves

with radii which decrease exponentially with distance below the surface. Figure 5.11 shows an example of these trajectories.

Figure 5.11: Trajectories of Water Particles in Short or Deep Water Waves

These trajectory radii, \( r \), decrease very fast with increasing distance below the free surface as a result of the negative value for \( z_1 \) in the exponential term \( e^{kz_1} \). This is illustrated here:

\[
\begin{align*}
  z &= 0 \quad \Rightarrow \quad r = \zeta_a \cdot e^{kz} = \zeta_a \cdot e^0 = 1.000 \cdot \zeta_a \\
  z &= -0.50\lambda \quad \Rightarrow \quad r = \zeta_a \cdot e^{kz} = \zeta_a \cdot e^{-\pi} = 0.043 \cdot \zeta_a \\
  z &= -\lambda \quad \Rightarrow \quad r = \zeta_a \cdot e^{kz} = \zeta_a \cdot e^{-2\pi} = 0.002 \cdot \zeta_a
\end{align*}
\]

One should remember from kinematics that the velocity of the water particles will have a constant magnitude in this case and that it is always directed tangentially to the trajectory circle.

**Accelerations**

The water particle accelerations follow directly from a differentiation of the velocity components - equations 5.57. This yields:

\[
\dot{u} = +\zeta_a \cdot \omega^2 \cdot \frac{\cosh k (h + z)}{\sinh kh} \cdot \sin (kx - \omega t)
\]
\[
\dot{w} = -\zeta_a \cdot \omega^2 \cdot \sinh k (h + z) \cdot \left( \cosh k h \right) \cos (kx - \omega t) \quad (5.65)
\]

Relative to the velocity components, the accelerations have amplitudes which have been multiplied by \( \omega \); their phases have been shifted by 90 degrees as well.

In deep water, the water particle accelerations are given by:

\[
\begin{align*}
\dot{u} &= +\zeta_a \omega^2 \cdot e^{kz} \cdot \sin (kx - \omega t) \\
\dot{w} &= -\zeta_a \omega^2 \cdot e^{kz} \cdot \cos (kx - \omega t) \quad \text{ (deep water)}
\end{align*}
\quad (5.66)
\]

It is wise to note that in a circular motion, the magnitude of the acceleration is constant and that it is always directed toward the center of the circle. This acceleration vector is then also always perpendicular to the velocity vector. It can be handy to remember this when computing forces on slender horizontal members in chapter 12.

### 5.2.4 Pressure

The pressure, \( p \), in first order wave theory follows from the linearized Bernoulli equation 5.26 so that:

\[
\frac{\partial \Phi_w}{\partial t} + \frac{p}{\rho} + gz = 0 \quad \text{or} \quad p = -\rho gz - \rho \frac{\partial \Phi_w}{\partial t} \quad (5.67)
\]

With the wave potential from equation 5.34 the expression for the linearized pressure becomes:

\[
\boxed{p = -\rho gz + \rho g \zeta_a \cdot \frac{\cosh k(h + z)}{\cosh kh} \cdot \cos(kx - \omega t)} \quad (5.68)
\]

For deep water (short waves), the linearized pressure becomes:

\[
p = -\rho gz + \rho g \zeta_a \cdot e^{kz} \cdot \cos(kx - \omega t) \quad \text{ (deep water)} \quad (5.69)
\]

The amplitude of the dynamic part of the pressure in short waves is \( \rho g \zeta_a \cdot e^{kz} \).

If the non-linear part (quadratic velocities) is retained, then the pressure in short waves becomes:

\[
p = -\rho gz + \frac{1}{2} \rho g \zeta_a^2 \omega^2 \cdot e^{2kz} + \rho g \zeta_a \cdot e^{kz} \cdot \cos(kx - \omega t) \quad \text{ (deep water)} \quad (5.70)
\]

in which use has been made of the trigonometric relation:

\[
\frac{1}{2} (u^2 + w^2) = \frac{1}{2} \zeta_a^2 \omega^2 \cdot e^{2kz} \cdot \{ \sin^2(kx - \omega t) + \cos^2(kx - \omega t) \} = \frac{1}{2} \zeta_a^2 \omega^2 \cdot e^{2kz}
\]

The first term in equation 5.70 is a hydrostatic part. The time-independent second term, the radiation pressure, causes the so-called second order wave drift loads on a structure; it will come up again in chapter 9. The harmonic third term causes the so-called first
order wave loads on a structure; its time-averaged contribution is zero. These loads will be treated in chapters 6 and 12.

In shallow water, the dynamic pressure behaves hydrostatically so that:

$$p = -\rho g z + \rho g \zeta_a \cos(kx - \omega t) \quad \text{(shallow water)}$$

5.2.5 Energy

The energy in a wave and the velocity at which this energy will be transported is of importance for, among others, the propulsion resistance caused by the waves generated by ships.

Wave Energy

Two types of energy are distinguished here: kinetic energy and potential energy.

![Figure 5.12: Determination of Kinetic Energy](image)

The kinetic energy, $K$, will be calculated as follows (see figure 5.12):

$$K = \int_{\text{volume}} \frac{1}{2} (u^2 + w^2) \cdot dm$$

$$= \frac{1}{2} \rho \int_{0}^{\lambda} \int_{-h}^{0} (u^2 + w^2) \cdot dz \cdot dx$$

$$= \frac{1}{2} \rho \int_{0}^{\lambda} \int_{0}^{h} (u^2 + w^2) \cdot dz \cdot dx + \frac{1}{2} \rho \int_{0}^{\lambda} \int_{0}^{\zeta} (u^2 + w^2) \cdot dz \cdot dx$$

$$= \frac{1}{2} \rho \int_{0}^{\lambda} \int_{0}^{\zeta} (u^2 + w^2) \cdot dz \cdot dx$$

The second term is approximately equal to:

$$\frac{1}{2} \rho \int_{0}^{\lambda} \int_{0}^{\zeta} (u^2 + w^2) \cdot dz \cdot dx \approx \frac{1}{2} \rho \int_{0}^{\lambda} \left\{ \int_{0}^{\zeta} dz \right\} (u^2 + w^2) \cdot dx$$
CHAPTER 5. OCEAN SURFACE WAVES

\[ K = \frac{1}{2} \rho \int_{0}^{\lambda} (u^2 + w^2) \cdot dx \]  

which is of second order - relative to the first term in equation 5.72 - and can be ignored. In a linearized form this equation then becomes:

\[ K = \frac{1}{2} \rho \int_{0}^{\lambda} (u^2 + w^2) \cdot dz \cdot dx \]  

A substitution of the expressions for \( u \) and \( w \) from equation 5.57 and using the dispersion relation of equation 5.42 and some algebra provides:

\[ K = \frac{1}{4} \rho g \zeta^2 \cdot \lambda \quad \text{per unit width (crest length)} \]
\[ = \frac{1}{4} \rho g \zeta_a^2 \quad \text{per unit horizontal sea surface area} \]  

Some trigonometric relations have been used to obtain these results:

\[ \int_{-h}^{0} \sinh^2 k (h + z) \cdot dz = \frac{1}{4k} \sinh 2kh - \frac{1}{2} h \]
\[ \int_{-h}^{0} \cosh^2 k (h + z) \cdot dz = \frac{1}{4k} \sinh 2kh + \frac{1}{2} h \]
\[ \int_{0}^{\lambda} \sin^2(kx - \omega t) \cdot dx = \int_{0}^{\lambda} \cos^2(kx - \omega t) \cdot dx = \frac{1}{2} \lambda \]  

Figure 5.13: Determination of Potential Energy

The potential energy, \( P \), will be calculated as follows (see figure 5.13):

\[ P = \frac{1}{2} \int_{0}^{\lambda} \rho g \zeta^2 \cdot dx \]
5.2. REGULAR WAVES

\[ = \frac{1}{2} \rho \zeta_a^2 \int_0^\lambda \cos^2(kx - \omega t) \cdot dx \]  
(5.77)

Some algebra results in:

\[ P = \frac{1}{4} \rho \zeta_a^2 \cdot \lambda \quad \text{per unit width (crest length)} \]
\[ = \frac{1}{4} \rho \zeta_a^2 \quad \text{per unit horizontal sea surface area} \]  
(5.78)

This has the same magnitude as the kinetic energy.

The total wave energy, \( E \), follows from equations 5.75 and 5.78, \( E = K + P \):

\[ E = \frac{1}{2} \rho \zeta_a^2 = \frac{1}{8} \rho g H^2 \quad \text{per unit horizontal sea surface area} \]  
(5.79)

Energy Transport or Power

The velocity at which the wave energy will be transported can be determined using figure 5.14. It shows a virtual vertical plane \( AA' \), which is perpendicular to the direction of propagation of the wave. An element in this plane (shaded in figure 5.14) is of unit width and has height \( dz \).

![Figure 5.14: Wave Energy Transport](image)

When the fluid passes through this plane element, a certain amount work \( dW \) will be delivered in a direction against the pressure \( p \) of the fluid.

The work done is force times distance:

\[ dW = \{ p \cdot 1 \cdot dz \} \cdot \{ u \cdot dt \} \]  
(5.80)

The average work done over one period, \( T \), or power is after linearization:
\[
\overline{W} = \frac{1}{T} \int_{t}^{t+T} \int_{-h}^{0} p \cdot u \cdot dz \cdot dt
\]  
(5.81)

which follows if one realizes that the contribution from the uppermost segment - the splash zone - can be neglected:

\[
\frac{1}{T} \int_{t}^{t+T} \int_{0}^{\zeta} p \cdot u \cdot dz \cdot dt \approx 0
\]  
(5.82)

Substituting 5.71 for the pressure and 5.57 for \( u \) results in:

\[
\overline{W} = -\frac{\rho k g^2 \zeta_a}{T \cdot \omega \cdot \sinh kh} \int_{t}^{t+T} \int_{-h}^{0} z \cdot \cosh k(h + z) \cdot \cos(kx - \omega t) \cdot dz \cdot dt
\]

\[
+ \frac{\rho k g^2 \zeta_a^2}{T \cdot \omega \cdot \sinh kh \cdot \cosh kh} \int_{t}^{t+T} \int_{-h}^{0} \cosh^2 k(h + z) \cdot \cos^2(kx - \omega t) \cdot dz \cdot dt
\]  
(5.83)

The first term is zero and, with the relations given in equations 5.76, the average work done or power becomes:

\[
\overline{W} = \frac{\rho g \zeta_a^2 \omega}{\sinh 2kh} \cdot \left( \frac{1}{4k} \sinh 2kh + \frac{1}{2} h \right)
\]  
(5.84)

With \( c = \omega/k \), this relation becomes:

\[
\overline{W} = \frac{1}{2} \rho g \zeta_a^2 \cdot \frac{c}{2} \cdot \left( 1 + \frac{2kh}{\sinh 2kh} \right)
\]  
(5.85)

or in terms of wave height:

\[
\overline{W} = \frac{1}{8} \rho g H^2 \cdot \frac{c}{2} \cdot \left( 1 + \frac{2kh}{\sinh 2kh} \right)
\]  
(5.86)

**Group Velocity**

The average work done per wave period in equation 5.85 can also be written as:

\[
\overline{W} = E \cdot c_g
\]  
(5.87)

with:

\[
E = \frac{1}{2} \rho g c_a^2 = \frac{1}{8} \rho g H^2
\]

\[
c_g = \frac{c}{2} \cdot \left( 1 + \frac{2kh}{\sinh 2kh} \right)
\]  
(5.88)

Then \( E \) is the energy of the waves per unit area and \( c_g \) is the velocity at which this energy will be transported in the waves, the wave group velocity.
5.2. REGULAR WAVES

In deep water, the wave group velocity is exactly half the phase velocity:

\[ c_g = \frac{c}{2} \quad \text{(deep water)} \quad (5.89) \]

In shallow water, the group velocity is identical to the phase velocity:

\[ c_g = c \quad \text{(shallow water)} \quad (5.90) \]

The characteristics of the wave phase velocity and the wave group velocity are shown in figure 5.15, obtained from [Newman, 1977].

Figure 5.15: Wave Phase and Group Velocity

The sequence of images shows a plane progressive wave system advancing into calm water. The water is darkened and the lower part of the water depth is not shown. The interval between the successive images is 0.25 seconds and \(\lambda/h \approx 2\). The wave energy is contained within the heavy diagonal lines and propagates with the group velocity. These wave group boundaries propagate slowly with time, due to dispersion. The position of a single wave crest is connected in successive images by the lighter line; this advances with the wave phase velocity. Each wave crest moves with the phase velocity, equal to twice the group velocity of the wave field boundaries. Thus each wave crest vanishes at the front end and, after the wave maker is turned off, arises from calm water at the back.
5.2.6 Relationships Summary

Figure 5.16 shows the relation between $\lambda$, $T$, $c$ and $h$ for a wide variety of conditions. Notice the boundaries $\lambda/h \approx 2$ and $\lambda/h \approx 20$ in this figure between short (deep water) and long (shallow water) waves.

![Diagram of relationships between $\lambda$, $T$, $c$ and $h$]

Figure 5.16: Relationships between $\lambda$, $T$, $c$ and $h$

Some examples of the effect of the water depth on the trajectories of individual water particles under a 100 meter length wave (at $\lambda/h = 50, 10$ and $1$) are presented in figure 5.17.

5.2.7 Shoaling Water

The theory in the previous sections has been worked out for water of constant depth. Attention in this section focuses on how waves change as they encounter varying water depths as they propagate. Some items - such as changes in the wave celerity and group speed - can be interfered from the theory already presented. Other wave properties including its wave height change too. Since the wave height (or amplitude) is so important for other wave-related phenomena, this section first discusses the influence of changing water depth on wave height.
5.2. REGULAR WAVES

Wave Height Changes

The relationship between wave height, $H$, and water depth, $h$, is embedded in the concept of conservation of energy transport through a vertical plane parallel to the wave crests. This is thus also the wave power per unit of crest length or what is sometimes called its energy flux, see equation 5.86.

By assuming that $W$ in this equation remains constant as the water depth changes, one can compute the wave height, $H_h$, in any water depth, $h$, as long as the wave height is known in some given water depth. It can be convenient for offshore and coastal engineering problems to work from (known) conditions in deep water, often denoted by a subscript "$\infty$". When one realizes that $k = 2\pi / \lambda$ - is purely a function of $h / \lambda$ or for that matter $h / \lambda_\infty$, one can determine - after quite some algebra - that:

$$K_{sh} = \frac{H_h}{H_\infty} = \sqrt{\frac{1}{\tanh kh \cdot \left(1 + \frac{2k h}{\sinh 2kh}\right)}}$$

(5.91)

in which:

- $K_{sh}$ = shoaling coefficient
- $H_\infty$ = wave height in deep water (m)
- $H_h$ = wave height in water of depth $h$ (m)
- $k$ = wave number at depth $h$ (rad/m)
- $h$ = chosen water depth (m)

If wave conditions are known only at some finite water depth, then one can first use equation 5.91 to evaluate $K_{sh}$ at that depth and then compute a new shoaling coefficient at the desired water depth. The known wave height can then be transformed to one at the new depth using the ratio of the shoaling coefficients.

Figure 5.18 shows how a wave changes height as it progresses into shallower water. The shoaling coefficient is actually plotted on the vertical axis of this figure (as wave height or velocity ratio) as a function of the dimensionless water depth $h / \lambda_\infty$ on the horizontal axis. A wave progressing into shallower water would move from left to right. One sees that the wave height first decreases a bit and then recovers. It is not until a wave gets into water too shallow to be of offshore interest that its height becomes larger than in deep water. Obviously the wave will ultimately break as the water depth becomes ever shallower. Only swimmers, surfboarders and coastal engineers enjoy waves breaking on the shore; sailors...
and offshore engineers want to avoid them instead. Indeed, this oceanographic detail is left to others.

**Water Motion Changes**

One may conclude from the above that the effects of water shoaling - at least on the wave height - are not that spectacular. How does the amplitude of the horizontal water velocity in the wave change? Since this amplitude at the sea surface can be given by (see equation 5.57):

$$u_a(h) = K_{sh} \cdot \frac{H_\infty}{2} \cdot \frac{\omega \cdot \cosh kh}{\sinh kh}$$  \hspace{1cm} (5.92)

Then the velocity ratio is given by:

$$\frac{u_a(h)}{u_a(\infty)} = K_{sh} \cdot \frac{\cosh kh}{\sinh kh}$$  \hspace{1cm} (5.93)

This is plotted in figure 5.18 as well; this increases more than does the wave height. Its importance will become obvious when survival loads on fixed offshore tower structures are being estimated. This is explained in chapter 13.

**Wave Refraction**

The wave period is the only property which does not change as a wave progresses through water of varying depth. Since the wave length does change, it is only logical, then, that the wave speed, $c = \lambda/T$, changes as well. Indeed, as the wave length becomes smaller (in shallower water) the wave must travel more slowly.

When now waves approach in such a way that their crests are not parallel to the depth contours, then one segment of each crest will be in shallower water than its neighboring segment. The segment in shallower water will move forward more slowly than that in deeper water. The wave crest - and thus its direction of propagation - will turn; this is
5.2. REGULAR WAVES

called refraction much like the similar phenomena with light in physics. The wave crests always turn in such a way that their crests become more parallel to the depth contours as they proceed into shallower water.

5.2.8 Wave Reflection and Diffraction

When a regular wave component meets a vertical wall perpendicular to its propagation direction - such as the side of a big ship - it is reflected and sent back where it came from with (ideally) an identical amplitude and speed. The water surface near the ship appears to move up and down - with twice the amplitude of the incoming wave - but there is no apparent wave progression. This describes a standing wave, which can be formulated by adding up two identical waves moving in opposite directions as given in equations 5.94.

\[ \zeta = \zeta_1 + \zeta_2 = \zeta_a \cos (kx - \omega t) + \zeta_a \cos (kx + \omega t) = 2\zeta_a \cos (kx) \cos (\omega t) \] (5.94)

The amplitude of this resulting wave is twice the amplitude of the two separate progressive wave components and the phase velocity becomes zero; see figures 5.19 and 5.20.

If the wave approaches the wall under an angle, then the above approach is still valid for its component perpendicular to the wall. The wave is reflected away from the wall at an angle in the same as light reflects from a flat mirror.

Wave diffraction is a process by which wave energy is propagated into a shadow zone - an area which is not directly in line with the approaching wave. Coastal engineers use
diffraction to predict waves in behind breakwaters in harbors. Since nearly all (fixed) breakwaters extend over the entire water depth, they work with two dimensional (in the horizontal plane) diffraction.

A large ship can also function as a floating breakwater. Indeed, persons being rescued from the sea are almost invariably picked up on the lee side of the ship - where the waves are lower.

Since the ship is floating, energy can now be transmitted to the shadow zone via the ends of the ship as well as under the ship. Wave diffraction even plays an essential role in the motion of the ship itself. Indeed, diffraction principles will be applied frequently when determining the motions of floating objects in waves. This topic is treated in chapters 6 through 9.

5.2.9 Splash Zone

Linear wave theory presented so far describes (formally) the water motion in the zone below the still water level. Indeed, as far as the theory is (strictly) concerned, the wave surface never departs from that level $z = 0$; this comes from the linearizations used. Such a limitation is especially inconvenient when splash zone hydromechanics becomes relatively more important. This can be the case when considering nonlinear phenomena in ship motions as in chapter 9 or when predicting survival loads on offshore tower structures in chapter 13. Another difficulty is that one usually finds that the wave crest elevation is generally higher than its corresponding trough is deep; the wave is actually somewhat asymmetrical, it does not have a perfect sinus-shape. Higher order wave theories were introduced to solve this latter problem.

Higher Order Wave Theories

In the past, there have been several attempts to alleviate the above limitation of linear theory in order - at least - to better describe the actual wave surface profile in mathematical terms, and - with luck - to describe the water kinematics up to the actual wave surface as well.

The theories one can find in the - primarily older - literature include the cynoidal theory and Stokes’ second, third and fifth order theories. All of these theories describe only a regular wave, by the way. They are all nonlinear which means that superposition - as described in the beginning of this chapter - cannot be used with them.

Generally speaking the increasing need for statistical information on the sea and a structure’s reaction to it, has led to the more widespread acceptance of linear methods in general and linear wave theory in particular. None of the above mentioned theories will be discussed here, therefore; many of them can still be found in other textbooks, however. Other ways to calculate the water motions in the splash zone have been found.

Profile Extension Methods

The first step is to realize that - especially for higher waves often used in design - the wave crest will be higher than the sinusoidal wave amplitude above the sea surface. The higher order wave theories, above, included this more or less automatically. When linear theory is used this must be done 'by hand’. It is usually assumed that the wave will extend 0.6 to
0.7 times its height above the still water level; two thirds or six tenths are commonly used fractions for this.

Once this has been taken care of, the next step is to adapt linear wave theory in some effective way. Remember that we can use the following equation to describe the horizontal water velocity component:

\[ u_a(z) = \frac{H}{2} \cdot \omega \cdot \frac{\cosh k(h + z)}{\sinh kh} \] (5.95)

This equation is derived from equation 5.57 by substituting \( \zeta_a = H/2 \), for use in the range \( 0 \leq z \leq -h \); it is valid for all water depths.

There are some simple profile extension methods available:

- **Extrapolation**
  This uses unaltered linear wave theory all the way from the sea bed to the actual water surface elevation or wave crest height; \( z \) is simply allowed to be positive in the formulas. Straightforward mathematical extension of linear theory in this way leads to an ‘explosion’ in the exponential functions so that the velocities become exaggerated within the wave crest; such results are generally considered to be too big or over-conservative.

- **Constant Extension**
  A second relatively simple method uses conventional linear wave theory below the mean sea level. Then the water velocity found at \( z = 0 \) is simply used for all values of \( z > 0 \). This is commonly used and is quite simple in a hand calculation. When the wave profile is below the still water level, then one simply works with linear theory up to that level, only.

- **Wheeler Profile Stretching**
  Profile stretching is a means to make the negative \( z \)-axis extend from the actual instantaneous water surface elevation to the sea bed. Many investigators have suggested mathematical ways of doing this, but Wheeler’s method presented here is the most commonly accepted, see [Wheeler, 1970]. He literally stretched the profile by replacing \( z \) (on the right hand side of equation 5.95 only!) by:

\[ z' = qz + h(q - 1) \]

in which:

\[ q = \frac{h}{h + \zeta} \] (5.96)

where:

- \( z' \) = a computational vertical coordinate (m): \(-h \leq z' \leq 0\)
- \( z \) = the actual vertical coordinate (m): \(-h \leq z \leq \zeta\)
- \( q \) = a dimensionless ratio (-)
- \( \zeta \) = the elevation of the actual water surface (m), measured along the \( z \)-axis
- \( h \) = the water depth to the still water level (m)

What one is doing, is really computing the water motion at elevation \( z' \) but using that water motion as if it were at elevation \( z \). Wheeler stretching is a bit cumbersome for use in a purely hand calculation. It can be implemented quite easily in a spreadsheet or other computer program, however. It is popular in practice, too.

One should note the following about the formulas in this section:

- If only a maximum velocity is needed, the time function can be neglected as has been done here. The time function plays no distinctive role in any of these methods.
• All $z'$-values used will be negative (or zero); the exponential functions 'behave themselves'.
• Each of these approaches degenerate to linear theory if it is applied at the still water level.
• The above formulations are universal in that they can be used in any water depth, at any point under a wave profile and at any particular phase within the wave (if desired).
• Since all of these stretching methods are based upon linear wave theory, they can all be used with irregular waves.

Users should be aware that use of deep water wave theory - with its simplifications - will lead to less than conservative results as the water depth decreases; this is the reason that the full theory is used above. Indeed, water motion amplitudes within a wave are never smaller than those predicted by deep water theory.

Comparison

A check out of the behavior of all of the above computational procedures in a typical situation is done here for the following condition: $H = 15$ meters, $T = 12$ seconds and $h = 40$ meters. These values (with a crest height which is $2/3 \cdot H$) yield $q = 40/50 = 0.80$. The computed results are shown in figure 5.21.

![Figure 5.21: Comparison of Computed Horizontal Water Velocities](image)

The following observations can be made:
• A consistently extrapolated Airy theory yields the largest values. This might well be expected.
• Similarly, a 'plain' Airy theory which simply neglects the wave crest entirely, yields a lower bound value.
• All methods presented give identical results at the sea bed - as they should.
• Wheeler stretching seems to yield attractive results in that they lie within the expected range and they are considerably less than the upper bound.
5.3 Irregular Waves

Now that the background of regular waves has been discussed, attention shifts to a more realistic image of the sea surface. Often the sea surface looks very confused; its image changes continuously with time without repeating itself. Both the wave length between two successive crests or troughs and the vertical distance between a crest and a trough vary continuously.

5.3.1 Wave Superposition

It was stated in the first lines of this chapter that it is possible to represent the irregular sea surface using a linear superposition of wave components. This will be demonstrated here. In fact, a superposition of only two components with identical directions but different speeds are needed to show this; see figure 5.22.

![Figure 5.22: Superposition of Two Uni-Directional Harmonic Waves](image)

A superposition of three components yields a more realistic record, even though these still come from one direction; see figure 5.23. A wave train such as this can easily be generated in a wave flume, for example. Since all of the energy travels in the same direction, the wave crests will be (theoretically) infinitely long; everything of interest can be observed in a single $x$-$z$ plane.

Note that for a realistic record of uni-directional irregular waves, a superposition of at least 15 or 20 components is required in practice if one is only interested in the mean value of an output. More components are handy if additional information such as statistical distributions are needed.

If a third dimension - direction - is added to this, then the sea (as seen from above) becomes even more realistic as is shown in figures 5.24 and 5.25. One sees from these figures that the length of the wave crests is now limited. This is a sign of the fact that wave energy is simultaneously propagating in several directions.

5.3.2 Wave Measurements

Most waves are recorded at a single fixed location. The simplest instrumentation can be used to simply record the water surface elevation as a function of time at that location. Since only a scalar, single point measurement is being made, the resulting record will yield no information about the direction of wave propagation.
CHAPTER 5. OCEAN SURFACE WAVES

Figure 5.23: Superposition of Three Uni-Directional Harmonic Waves

Figure 5.24: Superposition of Two Harmonic Waves with Different Directions
5.3. IRREGULAR WAVES

Figure 5.25: Superposition of Three Harmonic Waves with Different Directions

It is possible to measure (or to infer) wave directions at a point by using more sophisticated instruments to measure - for example - simultaneous horizontal water velocity components in two perpendicular directions and to correlate these measurements with the water surface time record.

Today it is quite common to measure wave heights from satellites using high frequency radar signals. Physically, the scatter of (or noise in) the reflected radar signal is a measure of the roughness of the surface reflecting it - in this case the sea. This noise can then be correlated with the wave height present on the sea surface.

Waves have occasionally been measured by stereo photography from above; it is pretty expensive to do. Such images have been used to study wave growth in a wind field and to determine the wave’s directional spreading.

5.3.3 Simple Statistical Analysis

Figure 5.26 shows a part of a simple time history of an irregular wave. When such a time history is available, then a simple analysis can be carried out to obtain statistical data from this record. The length of this time history should be at least 100 times the longest wave period, to obtain reliable statistical information. The analysis presented in this section can be carried out by hand.

Average Wave Period

The average wave period, \( \bar{T} \), can be found easily (by hand) from the average zero up-crossing period or from the average period of the wave crests or troughs. The simplest way to do this is to divide the record duration by the number of upward (or downward) zero-crossings found minus one.
Obtaining an average wave height requires more work. Successive wave heights are measured and classified in groups with intervals of, for instance, 0.5 meter. The number of wave heights in each group is then counted. These counts for each group are divided by the total number of wave heights to obtain the frequency quotients or the probability density function \( f(x) \). Finally these frequency quotients are added cumulatively to obtain the cumulative frequency quotients or the distribution function of wave heights \( F(x) \).

A numerical example of this approach is given in the table below.

<table>
<thead>
<tr>
<th>wave height intervals (m)</th>
<th>wave height average (m)</th>
<th>number of waves ( n )</th>
<th>frequency quotient ( f(x) )</th>
<th>cumulative frequency quotient ( F(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25-0.75</td>
<td>0.5</td>
<td>15</td>
<td>0.100</td>
<td>0.100</td>
</tr>
<tr>
<td>0.75-1.25</td>
<td>1.0</td>
<td>30</td>
<td>0.200</td>
<td>0.300</td>
</tr>
<tr>
<td>1.25-1.75</td>
<td>1.5</td>
<td>55</td>
<td>0.367</td>
<td>0.667</td>
</tr>
<tr>
<td>1.75-2.25</td>
<td>2.0</td>
<td>21</td>
<td>0.140</td>
<td>0.807</td>
</tr>
<tr>
<td>2.25-2.75</td>
<td>2.5</td>
<td>14</td>
<td>0.093</td>
<td>0.900</td>
</tr>
<tr>
<td>2.75-3.25</td>
<td>3.0</td>
<td>9</td>
<td>0.060</td>
<td>0.960</td>
</tr>
<tr>
<td>3.25-3.75</td>
<td>3.5</td>
<td>5</td>
<td>0.033</td>
<td>0.993</td>
</tr>
<tr>
<td>3.75-4.25</td>
<td>4.0</td>
<td>1</td>
<td>0.007</td>
<td>1.000</td>
</tr>
<tr>
<td>total</td>
<td></td>
<td>150</td>
<td></td>
<td>1.000</td>
</tr>
</tbody>
</table>

The probability density function of wave heights, \( f(x) \), is given in figure 5.27-a as a histogram. The associated distribution function of wave heights, \( F(x) \), is given in figure 5.27-b. As one uses even more waves to determine these functions, the functions converge to a stable form.

Statistical information can be obtained from the probability density function \( f(x) \). For example, the probability that the wave height, \( \tilde{H}_w \), exceeds a certain threshold value, \( a \), in this record is given by:

\[
P \{ \tilde{H}_w > a \} = \int_a^\infty f(x) \cdot dx
\]  

(5.97)
5.3. IRREGULAR WAVES

Figure 5.27: Probability Density and Distribution Functions of Wave Heights

The table shows that, if $a = 3.25$ meter, the probability of finding a wave higher than that threshold value is given by this integral (or in this case):

$$P \left\{ H_w > 3.25 \right\} = \frac{5 + 1}{150} = 0.04 \quad \text{or} \quad P \left\{ H_w > 3.25 \right\} = 0.033 + 0.007 = 0.04 \quad (5.98)$$

Mean Wave Height

The mean wave height can be found very easily using either figure as follows:

$$\overline{H} = \frac{0.5 \cdot 15 + 1.0 \cdot 30 + 1.5 \cdot 55 + 2.0 \cdot 21 + 2.5 \cdot 14 + 3.0 \cdot 9 + 3.5 \cdot 5 + 4.0 \cdot 1}{150} = 1.64 \text{ m}$$

or from $f(x)$:

$$\overline{H} = 0.5 \cdot 0.100 + 1.0 \cdot 0.200 + 1.5 \cdot 0.367 + 2.0 \cdot 0.140 + 2.5 \cdot 0.093 + 3.0 \cdot 0.060 + 3.5 \cdot 0.033 + 4.0 \cdot 0.07 = 1.64 \text{ m}$$

Significant Wave Height

The so-called significant wave height $H_{1/3}$, defined as the average of the highest 1/3 of the waves in the record. Thus, in this case:

$$H_{1/3} = \frac{2.0 \cdot 21 + 2.5 \cdot 14 + 3.0 \cdot 9 + 3.5 \cdot 5 + 4.0 \cdot 1}{50} = 2.51 \text{ m}$$

or from $f(x)$:

$$H_{1/3} = \frac{2.0 \cdot 0.140 + 2.5 \cdot 0.093 + 3.0 \cdot 0.060 + 3.5 \cdot 0.033 + 4.0 \cdot 0.07}{1/3} = 2.51 \text{ m}$$

The significant wave height, $H_{1/3}$, plays an important role in many practical applications of wave statistics. Often there is a fair correlation between the significant wave height and a visually estimated wave height. This comes, perhaps, because higher waves make more of an impression on an observer than the smallest ones do.
5.3.4 More Complete Record Analysis

Now a more complete record analysis will be given; it is only possible using a computer and a digitized time record of the wave surface elevation.

Record and Sampling

![Figure 5.28: Registration and Sampling of a Wave](image)

Consider an irregular wave record, \( \zeta(t) \), as given in figure 5.28. As already has been mentioned, the length of the record should be at least 100 times the longest observed period of the waves in this record. The record will be sampled at a large number, \( N \), equal intervals, \( \Delta t \). In practice, one might make a record of about 15 to 20 minutes, spaced every half second. Unless there is a very long swell in the record, this is just long enough to capture enough waves, but still short enough to avoid extra influences such as result from tidal level changes. The average level of these \( N \) observations (relative to some arbitrary level) can be found easily and used to define the still water level. Once the elevations have been corrected to this level, one has a set of \( N \) vertical displacements, \( \zeta_n \), relative to the still water level.

Standard Deviation

The so-called standard deviation \( \sigma \) of the water level \( \zeta(t) \) follows from:

\[
\sigma = \sqrt{\frac{1}{N-1} \sum_{n=1}^{N} \zeta_n^2} \quad \text{(standard deviation)} \tag{5.99}
\]

Notice that \( \sigma \) relates to the instantaneous water surface elevations, instead of crest-to-trough wave heights as was used before in the simple statistical analysis.

This standard deviation \( \sigma \) - also referred to as Root Mean Square (RMS) value - is related to the significant wave amplitude, \( \zeta_{a1/3} \), and the significant wave height, \( H_{1/3} \), by:

\[
\begin{align*}
\zeta_{a1/3} &= 2 \cdot \sigma \quad \text{(significant wave amplitude)} \\
H_{1/3} &= 4 \cdot \sigma \quad \text{(significant wave height)} \tag{5.100}
\end{align*}
\]
5.3. IRREGULAR WAVES

Figure 5.29: Normal or Gaussian Distribution with $\sigma = 1$

Gaussian Water Level Distribution

The above series of water levels, $\zeta$, can be analyzed statistically. It has been found to fit a Gaussian distribution or normal distribution quite well; see figure 5.29. This distribution - provided that the $\zeta$-values have a mean value equal to zero - is given by:

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{- \left( \frac{x}{\sigma \sqrt{2}} \right)^2 \right\} \quad \text{(Gaussian or normal distribution)} \quad (5.101)$$

in which $x$ is the variable being studied and $\sigma$ is its standard deviation. With this distribution, the probability that the water level, $\zeta$, exceeds a chosen threshold value, $a$, can be calculated using:

$$P \{ \zeta > a \} = \int_a^\infty f(x) \cdot dx$$

$$= \frac{1}{\sigma \sqrt{2\pi}} \int_a^\infty \exp \left\{- \left( \frac{x}{\sigma \sqrt{2}} \right)^2 \right\} \cdot dx \quad (5.102)$$

This integral calculates the area under the curve to the right of the chosen value, $a$. Some important values of $a$ and corresponding probabilities, $P \{ \zeta > a \}$, are given in the table below.
A Gaussian or normal distribution with $\sigma = 1$ is shown in figure 5.29. This distribution has its inflection points at $x = \pm \sigma = \pm 1$ and its value at $x = 0$ is $f(0) = 1/(\sigma \sqrt{2\pi}) = 0.399$.

Rayleigh Wave Amplitude Distribution

If the range of frequencies in a wave field is not too large, one speaks of a narrow banded frequency spectrum. Luckily, waves - a sea or a swell - generally satisfy this condition quite well. If this is the case - and the water surface elevation is a Gaussian distribution - then the wave amplitude statistics will obey a Rayleigh distribution.

This Rayleigh distribution is given by:

$$f(x) = \frac{x}{\sigma^2} \cdot \exp \left\{ - \left( \frac{x}{\sigma \sqrt{2}} \right)^2 \right\}$$

(Rayleigh distribution) \hspace{1cm} (5.103)

in which $x$ is the variable being studied and $\sigma$ is its standard deviation.

With this Rayleigh distribution, the probability that the wave amplitude, $\zeta_a$, exceeds a chosen threshold value, $a$, can be calculated using:

$$P\{\zeta_a > a\} = \int_a^\infty f(x) \cdot dx$$

$$= \frac{1}{\sigma^2} \int_a^\infty x \cdot \exp \left\{ - \left( \frac{x}{\sigma \sqrt{2}} \right)^2 \right\} \cdot dx$$

$$= \exp \left\{ - \frac{a^2}{2\sigma^2} \right\}$$

(5.104)

These probabilities for wave crest elevations can also be expressed in terms of the crest-to-trough wave heights $H_w$:

$$P\{H_w > H\} = \exp \left\{ -2 \left( \frac{H}{H_{1/3}} \right)^2 \right\}$$

(5.105)

in which $P\{H_w > H\}$ is the chance that any individual wave height $H$ is exceeded in a wave field characterized by a significant wave height, $H_{1/3} = 4\sigma$.

The Rayleigh probability density function is illustrated in figure 5.30-a; figure 5.30-b shows the probability of exceeding in percentages, $P(H)$, in relation to the wave height ratio, $H/H_{1/3}$, on a logarithmic scale. Notice that all values in figures 5.30-a and 5.30-b are now positive. Indeed, if one chooses $H = 0$ in equation 5.105, then one finds that the probability that it is exceeded is exactly 1.
5.4. WAVE ENERGY SPECTRA

Maximum Wave Height

It is often desirable to make a statistically-based guess as to the highest wave that can be expected in a storm. One reasoning is to assume that the chance that this wave will be exceeded is zero and to put this value into equation 5.105 and then to solve for $H$. One must do this only once to find that $H = \infty$ - not too practical a result!

Instead of this, more pragmatic engineers often choose the wave height that will be exceeded (on the average) once in every 1000 (storm) waves as a maximum. This may seem arbitrary, (and it is!), but they reason that it will take at least 3 hours for 1000 waves to pass by and that by that time the worst peak of the storm will probably be past, anyway. This reasoning leads - with equation 5.105 - to a quick rule of thumb that:

$$\exp \left\{ -2 \left( \frac{H_{\text{max}}}{H_{1/3}} \right)^2 \right\} = \frac{1}{1000} \quad \text{or:} \quad H_{\text{max}} = 1.86 \cdot H_{1/3}$$

(5.106)

in which $H_{\text{max}}$ is the maximum expected wave height in this 3 hours storm.

5.4 Wave Energy Spectra

Little attention has been given to wave periods so far. Periods are the reciprocal of frequency and become more apparent as a by-product of the wave frequency spectra to be discussed in this section.
5.4.1 Basic Principles

Since an irregular wave can be seen as the superposition of a series of sinusoidal waves, it is only logical to study the frequency characteristics of such an irregular signal using Fourier series analysis. To do this, one selects a time record segment containing many waves. Indeed, one assumption implicit in this analysis - see appendix C - is that the signal being studied repeats itself after each (long) interval. A wave record does not do this exactly, but that little detail is neglected in this case. The wave elevation (in the time domain) of a long-crested irregular sea, propagating along the positive \(x\) axis, can be written as the sum of a large number of regular wave components (in the frequency domain):

\[
\zeta(t) = \sum_{n=1}^{N} \zeta_{a_n} \cos(k_n x - \omega_n t + \varepsilon_n)
\]  

(5.107)

in which, for each component, \(n\):

\[
\begin{align*}
\zeta_{a_n} & = \text{wave amplitude component (m)} \\
\omega_n & = \text{circular frequency component (rad/s)} \\
k_n & = \text{wave number component (rad/m)} \\
\varepsilon_n & = \text{random phase angle component (rad)}
\end{align*}
\]

A Fourier series analysis carried out at one location would not indicate anything about \(k\); this is location-dependent. This is not a problem in our case, since \(k_n\) and \(\omega_n\) are related by the known dispersion relation already given earlier in this chapter. The Fourier series will thus yield a set of values for \(\zeta_{a_n}\) and \(\varepsilon_n\), each associated with its own \(\omega_n\). If enough Fourier series terms are included, the entire time record at that point can be reproduced using this set of values. In practice, however, one is not really interested in the exact water level at some time \(t\) (which is already history!). It is sufficient to know only the statistical properties - now in terms of both frequency and amplitude - of the signal. This means that the \(\varepsilon_n\) can be discarded.

5.4.2 Energy Density Spectrum

Suppose a time history, as given in figure 5.28, of the wave elevation during a sufficient long but arbitrary period:

\[
\tau = N \cdot \Delta t
\]

The instantaneous wave elevation is supposed to has a Gaussian distribution and zero mean. The amplitudes \(\zeta_{a_n}\) can be obtained by a Fourier analysis of the signal. However, for each little time shift of the time history one will find a new series of amplitudes \(\zeta_{a_n}\). Luckily, a mean square value of \(\zeta_{a_n}\) can be found: \(\overline{\zeta_{a_n}^2}\).

When \(\zeta(t)\) is an irregular signal without prevailing frequencies, the average values \(\overline{\zeta_{a_n}^2}\) close to \(\omega_n\) will not change much as a function of the frequency; \(\overline{\zeta_{a_n}^2}\) is a continuous function.
The variance \( \sigma_\zeta^2 \) of this signal equals:

\[
\sigma_\zeta^2 = \frac{1}{N} \sum_{n=1}^{N} \zeta_n^2 = \frac{1}{N} \Delta t \sum_{n=1}^{N} \zeta_n^2 \cdot \Delta t
\]

\[
= \frac{1}{\tau} \int_0^\tau \zeta^2(t) \cdot dt = \frac{1}{\tau} \int_0^\tau \left\{ \sum_{n=1}^{N} \zeta_n \cos(\omega_n t - k_n x + \varepsilon_n) \right\}^2 \cdot dt
\]

\[
= \sum_{n=1}^{N} \frac{1}{2} \zeta_n^2
\]

(5.108)

The wave amplitude \( \zeta_n \) can be expressed in a wave spectrum \( S_\zeta(\omega_n) \), which expression is defined by:

\[
S_\zeta(\omega_n) \cdot \Delta \omega = \frac{\omega_n + \Delta \omega}{\omega_n} \sum_{\omega_n} \frac{1}{2} \zeta_n^2(\omega)
\]

(5.109)

where \( \Delta \omega \) is a constant difference between two successive frequencies. Multiplied with \( \rho g \), this expression is the energy per unit area of the waves (see equation 5.79) in the frequency interval \( \Delta \omega \) see figure 5.31.

Figure 5.31: Definition of Spectral Density

Letting \( \Delta \omega \to 0 \), the definition of the wave energy spectrum \( S_\zeta(\omega) \) becomes:

\[
S_\zeta(\omega_n) \cdot d\omega = \frac{1}{2} \zeta_n^2
\]

(5.110)

and the variance \( \sigma_\zeta^2 \) of the water surface elevation is simply equal to the area under the spectrum:

\[
\sigma_\zeta^2 = \int_0^\infty S_\zeta(\omega) \cdot d\omega
\]

(5.111)
Figure 5.32 gives a graphical interpretation of the meaning of a wave spectrum and how it relates to the waves. The irregular wave history, $\zeta(t)$ in the time domain at the lower left hand part of the figure can be expressed via Fourier series analysis as the sum of a large number of regular wave components, each with its own frequency, amplitude and phase in the frequency domain. These phases will appear to be rather random, by the way. The value $\frac{1}{2}c^2_{a}(\omega)\Delta\omega$ - associated with each wave component on the $\omega$-axis - is plotted vertically in the middle; this is the wave energy spectrum, $S_{\zeta}(\omega)$. This spectrum, $S_{\zeta}(\omega)$, can be described nicely in a formula; the phases cannot and are usually thrown away.

Figure 5.32: Wave Record Analysis

**Spectrum Axis Transformation**

When wave spectra are given as a function of frequency in Hertz ($f = 1/T$) instead of $\omega$ (in radians/second), they have to be transformed. The spectral value for the waves, $S_{\zeta}(\omega)$, based on $\omega$, is not equal to the spectral value, $S_{\zeta}(f)$, based on $f$. Because of the requirement that an equal amount of energy must be contained in the corresponding frequency intervals $\Delta\omega$ and $\Delta f$, it follows that:

$$|S_{\zeta}(\omega) \cdot d\omega = S_{\zeta}(f) \cdot df| \quad \text{or:} \quad S_{\zeta}(\omega) = \frac{S_{\zeta}(f)}{d\omega / df}$$ \hspace{1cm} (5.112)

The relation between the frequencies is:

$$\omega = 2\pi \cdot f \quad \text{or:} \quad \frac{d\omega}{df} = 2\pi$$ \hspace{1cm} (5.113)
5.4. WAVE ENERGY SPECTRA

Then the wave spectrum on an $\omega$-basis is:

$$S_\zeta(\omega) = \frac{S_\zeta(f)}{2\pi}$$  \hspace{1cm} (5.114)

An example of a spectrum transformation is given in figure 5.33, in which it is obvious that the ratio between the corresponding frequencies is $1/(2\pi)$ while the ratio between the corresponding spectral values is $2\pi$. The areas of both spectra (significant amplitudes) remain equal and the spectral moments provide equal average periods.

**Wave Height and Period**

Relationships with statistics can be found from computing the moments of the area under the spectrum with respect to the vertical axis at $\omega = 0$. If $m$ denotes a moment, then $m_{n\zeta}$ denotes the $n^{th}$ order moment given in this case by:

$$m_{n\zeta} = \int_0^\infty \omega^n \cdot S_\zeta(\omega) \cdot d\omega$$  \hspace{1cm} (5.115)

This means that $m_{0\zeta}$ is the area under the spectral curve, $m_{1\zeta}$ is the first order moment (static moment) of this area and $m_{2\zeta}$ is the second order moment (moment of inertia) of this area.

As has already been indicated, $m_{0\zeta}$ is an indication of the variance squared, $\sigma_\zeta^2$, of the water surface elevation. Of course this $m_{0\zeta}$ can also be related to the various wave amplitudes and heights:

$$\sigma_\zeta = RMS = \sqrt{m_{0\zeta}}$$  \hspace{1cm} (Root Mean Square of the water surface elevation)
\[
\begin{align*}
\zeta_{a_{1/3}} &= 2 \cdot \sqrt{m_0} & \text{(significant wave amplitude)} \\
H_{1/3} &= 4 \cdot \sqrt{m_0} & \text{(significant wave height)}
\end{align*}
\]

(5.16)

Characteristic wave periods can be defined from the spectral moments:

\[
m_1\zeta = \omega_1 \cdot m_0\zeta \quad \text{with } \omega_1 \text{ is spectral centroid}
\]

\[
m_2\zeta = \omega_2^2 \cdot m_0\zeta \quad \text{with } \omega_2 \text{ is spectral radius of inertia}
\]

(5.17)

as follows:

\[
\begin{align*}
T_1 &= 2\pi \cdot \frac{m_0\zeta}{m_1\zeta} & \text{(mean centroid wave period)} \\
T_2 &= 2\pi \cdot \sqrt{\frac{m_0\zeta}{m_2\zeta}} & \text{(mean zero-crossing wave period)}
\end{align*}
\]

(5.18)

The mean zero-crossing period, \( T_2 \), is sometimes indicated by \( T_z \). One will often find the period associated with the peak of the spectrum, \( T_p \), in the literature as well.

**Rayleigh Distribution**

Expressed in terms of \( m_{0x} \), the Rayleigh distribution of amplitudes \( x_a \) is given by:

\[
f(x) = \frac{x}{m_{0x}} \cdot \exp\left\{ -\frac{x^2}{2 \cdot m_{0x}} \right\} \quad \text{(Rayleigh distribution)}
\]

(5.19)

in which \( x \) is the variable being studied and \( m_{0x} \) is the area under the spectral curve.

With this distribution, the probability that the amplitude, \( x_a \), exceeds a chosen threshold value, \( a \), can be calculated using:

\[
P\{x_a > a\} = \int_a^\infty f(x) \cdot dx
\]

\[
= \frac{1}{m_{0x}} \left[ \int_a^\infty x \cdot \exp\left\{ -\frac{x^2}{2 \cdot m_{0x}} \right\} \cdot dx \right]
\]

\[
= \exp\left\{ -\frac{a^2}{2 \cdot m_{0x}} \right\}
\]

(5.20)

It is obvious here that \( \exp\{-u\} \) means \( e^{-u} \).

As an example for waves, the probability that the wave height, \( H_w \), in a certain sea state exceeds the significant wave height, \( H_{1/3} \), is found by:

\[
P\{H_w > H_{1/3}\} = P\{\zeta_a > \zeta_{a_{1/3}}\}
\]

\[
= \exp\left\{ -\frac{\zeta_{a_{1/3}}^2}{2 \cdot m_0} \right\}
\]

\[
= e^{-2} \approx 0.135 \approx \frac{1}{1e}
\]

(5.21)
Wave Record Length

An important problem in the conduct of irregular wave measurements is the required total duration of the measured time histories to obtain proper spectral shapes and statistical values. This duration is presented by the total number of wave cycles, \( N \).

![Figure 5.34: Effect of Wave Record Length](source)

Figure 5.34 shows an example of the fluctuations in several characteristic values derived from calculated spectra based on records containing varying numbers of wave cycles. Using a wave record of a sea with a period \( T_1 \approx 6 \text{ s} \), calculations were carried out for various record lengths, expressed in a varied number of \( N \) cycles. The ratios of the calculated characteristics \( H_{1/3}, T_1 \) and \( T_2 \) for \( N \) cycles and those for a very large number of cycles are presented in the figure. It can be seen that these ratios become more or less constant for \( N > 50 \). This value can be considered as a rough standard for the absolute minimum required number of cycles. It can be seen from the figure that a much larger number is required to produce results approaching the base line, where this ratio becomes 1.0.

According to the 17th International Towing Tank Conference in 1984, \( N = 50 \) should be taken as a lower limit. Larger values are to be preferred and it is more usual to take \( N = 100 \) as the usual standard. A number \( N = 200 \) or above is considered excellent practice by the I.T.T.C. In practice, a record length equal to 100 times the largest expected single wave period in the irregular waves is often used as a safe standard. This means a wave record length of about 15 to 20 minutes.

As well as influencing the outcome of spectral analyses in the frequency domain, the number of wave cycles, \( N \), also affects the validity of statistical quantities calculated in the time domain such as mean values, probability densities and distributions of extreme values.

### 5.4.3 Standard Wave Spectra

Just as for wave height statistics, investigators have attempted to describe a wave frequency spectrum in a standard form. Two important ones often found in the literature are described here. The mathematical formulations of these normalized uni-directional wave
energy spectra are based on two parameters: the significant wave height, $H_{1/3}$, and average wave periods $\bar{T}$, defined by $T_1$, $T_2$ or $T_p$:

$$S_c(\omega) = H_{1/3}^2 \cdot f(\omega, \bar{T})$$ \hspace{1cm} (5.122)

Note that this definition means that the spectral values are proportional to the significant wave height squared; in other words $S_c(\omega) / H_{1/3}^2$ is a function of $\omega$ and $\bar{T}$ only.

**Bretschneider Wave Spectra**

One of the oldest and most popular wave spectra was given by Bretschneider. It is especially suited for open sea areas. It is given mathematically by:

$$S_c(\omega) = \frac{173 \cdot H_{1/3}^2}{T_1^4} \cdot \omega^{-8} \cdot \exp \left\{ \frac{-692}{T_1^4} \cdot \omega^{-4} \right\}$$ \hspace{1cm} (5.123)

For non-truncated mathematically defined spectra of this type, the theoretical relations between the characteristic periods are listed below:

$$
\begin{align*}
T_1 &= 1.086 \cdot T_2 = 0.772 \cdot T_p \\
0.921 \cdot T_1 &= T_2 = 0.711 \cdot T_p \\
1.296 \cdot T_1 &= 1.407 \cdot T_2 = T_p
\end{align*}
$$ \hspace{1cm} (5.124)

Another name of this wave spectrum is Modified Two-Parameter Pierson-Moskowitz Wave Spectrum. This formulation was accepted by the 2nd International Ship Structures Congress in 1967 and the 12th International Towing Tank Conference in 1969 as a standard for seakeeping calculations and model experiments. This is reason why this spectrum is also called the ISSC Wave Spectrum or the ITTC Wave Spectrum.

The original One-Parameter Pierson-Moskowitz Wave Spectrum for fully developed seas can be obtained by using a fixed relation between the significant wave height and the average wave period in the Bretschneider definition:

$$
\begin{align*}
T_1 &= 3.86 \cdot \sqrt{H_{1/3}} \quad \text{and} \quad T_2 = 3.56 \cdot \sqrt{H_{1/3}}
\end{align*}
$$ \hspace{1cm} (5.125)

In reality a measured spectral form differs from theoretical formulations which give only a mean distribution. Figure 5.35 shows a comparison between a measured wave spectrum and the corresponding Bretschneider (Pierson-Moskowitz) wave spectrum during a storm in the Atlantic Ocean on 4 February 1979.

**JONSWAP Wave Spectra**

In 1968 and 1969 an extensive wave measurement program, known as the Joint North Sea Wave Project (JONSWAP) was carried out along a line extending over 100 miles into the North Sea from Sylt Island. Analysis of the data - in which the TU Delft participated - yielded a spectral formulation for fetch-limited (or coastal) wind generated seas.

The following definition of a Mean JONSWAP wave spectrum is advised by the 17th ITTC in 1984 for fetch limited situations:

$$S_c(\omega) = \frac{320 \cdot H_{1/3}^2}{T_p^4} \cdot \omega^{-5} \cdot \exp \left\{ \frac{-1950}{T_p^4} \cdot \omega^{-4} \right\} \cdot \gamma^A$$ \hspace{1cm} (5.126)
5.4. WAVE ENERGY SPECTRA

Figure 5.35: Comparison of a Measured and a Normalised Wave Spectrum

with:

\[ \gamma = 3.3 \quad \text{(peakedness factor)} \]

\[ A = \exp \left\{ - \left( \frac{\omega}{\omega_p} - \frac{1}{\sigma \sqrt{2}} \right)^2 \right\} \]

\[ \omega_p = \frac{2\pi}{T_p} \quad \text{(circular frequency at spectral peak)} \]

\[ \sigma = \begin{cases} 0.07 & \text{if } \omega < \omega_p \text{ then: } \\ 0.09 & \text{if } \omega > \omega_p \text{ then: } \end{cases} \]

Taking \( \gamma^A = 1.522 \) results in the formulation of the Bretschneider wave spectrum with the peak period \( T_p \).

For non-truncated mathematically defined JONSWAP spectra, the theoretical relations between the characteristic periods are listed below:

\[ T_1 = 1.073 \cdot T_2 = 0.834 \cdot T_p \]

\[ 0.932 \cdot T_1 = T_2 = 0.777 \cdot T_p \]

\[ 1.199 \cdot T_1 = 1.287 \cdot T_2 = T_p \quad (5.127) \]

Sometimes, a third free parameter is introduced in the JONSWAP wave spectrum by varying the peakedness factor \( \gamma \).

Wave Spectra Comparison

Figure 5.36 compares the Bretschneider and mean JONSWAP wave spectra for three sea states with a significant wave height, \( H_{1/3} \), of 4 meters and peak periods, \( T_p \), of 6, 8 and
10 seconds, respectively.

![Figure 5.36: Comparison of Two Spectral Formulations](image)

The figure shows the more pronounced peak of the JONSWAP spectrum.

**Directional Spreading**

A cosine-squared rule is often used to introduce directional spreading to the wave energy. When this is done, the unidirectional wave energy found in the previous sections is scaled as in the following formula:

\[
S_z(\omega, \mu) = \left\{ \frac{2}{\pi} \cdot \cos^2 (\mu - \bar{\mu}) \right\} \cdot S_z(\omega)
\]

with:

\[-\frac{\pi}{2} \leq (\mu - \bar{\mu}) \leq +\frac{\pi}{2}\]

in which \(\bar{\mu}\) is the dominant wave direction. A comparison of a measured wave directionality with this directional spreading is given in figure 5.37.

Because the directionality function in this theory is a scalar, the form of the spectrum along each direction is the same; only its intensity varies as a function of direction. At sea, this distribution depends on the local weather situation at that moment (for the wind waves or sea) as well as on the weather in the whole ocean in the recent past (for any swell component). Deviations from the theoretical distribution will certainly appear when, for instance, when sea and swell travel in quite different directions.

**5.4.4 Transformation to Time Series**

Especially when solving non-linear problems - such as in chapters 9 or 12, for example - one often needs a deterministic time record of water levels which has the statistical properties
one would associate with a known spectrum. This is often referred to as the inverse problem in wave statistics. (It is at moments such as this that one is sorry that the phase information from the Fourier series analysis - which yielded the spectrum in the first place - has been thrown away.)

Luckily, there is no real need to reproduce the input time record exactly; what is needed is a record which is only statistically indistinguishable from the original signal. This means that it must have the same energy density spectrum as the original. This is done by filling in all the necessary constants in equation 5.107 which is repeated here for convenience.

\[ \zeta(t) = \sum_{n=1}^{N} \zeta_{an} \cos(k_n x - \omega_n t + \varepsilon_n) \]

The desired output - whether it is the water surface elevation as in this equation or anything else such as the horizontal component of the water particle velocity 17 meters below the sea surface - depends upon three variables for each chosen frequency component, \( \omega_n \): \( k_n \), \( \varepsilon_n \) and an amplitude such as \( \zeta_{an} \).

Successive \( \omega_n \) values are generally chosen at equally spaced intervals \( \Delta \omega \) along the frequency axis of the given spectrum. Note that with a constant frequency interval, \( \Delta \omega \), this time history repeats itself after \( 2\pi / \Delta \omega \) seconds.

The amplitudes, \( \zeta_{an} \), can be determined knowing that the area under the associated segment of the spectrum, \( S_\zeta(\omega) \cdot \Delta \omega \), is equal to the variance of the wave component. Equation 5.109 is adapted a bit for this so that:

\[ \zeta_{an} = 2 \sqrt{S_\zeta(\omega) \cdot \Delta \omega} \]  

(5.129)

The wave numbers, \( k_n \), can be computed from the chosen frequency \( \omega_n \) using a dispersion relationship.
When obtaining the wave spectrum $S_\omega(\omega)$ from the irregular wave history, the phase angles $\varepsilon_n$ have been thrown away. New $\varepsilon_n$-values must be selected from a set of uniformly distributed random numbers in the range $0 \leq \varepsilon_n < 2\pi$. While $\varepsilon_n$ is needed to generate the time record - and they may not all be set equal to zero! - the exact (randomly selected) $\varepsilon_n$ do not influence the record’s statistics. Looked at another way: By choosing a new random set of $\varepsilon_n$ values, one can generate a new, statistically identical but in detail different time record.

This procedure is illustrated by extending figure 5.32 as shown below in figure 5.38. Note that the original (on the left in the figure) and the newly obtained wave history (on the right hand part of the figure) differ because different phase angles have been used. However, they contain an equal amount of energy and are statistically identical.

Directional spreading can be introduced as well by breaking each frequency component up into a number of directional components. One should not underestimate the (extra) computational effort however.

### 5.5 Wave Prediction and Climatology

In 1805, the British Admiral Sir Francis Beaufort devised an observation scale for measuring winds at sea. His scale measures winds by observing their effects on sailing ships and waves. Beaufort’s scale was later adapted for use on land and is still used today by many weather stations. A definition of this Beaufort wind force scale is given in figure 5.39.

The pictures in figure 5.40 give a visual impression of the sea states in relation to Beaufort’s scale. Storm warnings are usually issued for winds stronger than Beaufort force 6.
<table>
<thead>
<tr>
<th>Beaufort Number</th>
<th>Knots</th>
<th>miles</th>
<th>meters</th>
<th>km</th>
<th>Wind Press. N/m²</th>
<th>Beaufort description for square rigged ships 1806</th>
<th>Racing Sailor's description (C.A. Marchay, 1964)</th>
<th>U.S. Weather Service description</th>
<th>Dutch KMI description</th>
<th>Beaufort Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.00</td>
<td>Calm</td>
<td>Windstil</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>0.5</td>
<td>0.53</td>
<td>Just Steerage Way</td>
<td>Boredom</td>
<td>Light air</td>
<td>zwakke</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>2.1</td>
<td>2.37</td>
<td>1-3 knots close hauled</td>
<td>Mild pleasure</td>
<td>Light breeze</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>10</td>
<td>12</td>
<td>3.6</td>
<td>3.81</td>
<td>4-5 knots close hauled</td>
<td>Pleasure</td>
<td>Gentle breeze</td>
<td>matige</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
<td>16</td>
<td>18</td>
<td>5.7</td>
<td>5.94</td>
<td>6-7 knots close hauled</td>
<td>Great Pleasure</td>
<td>Moderate breeze</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>17</td>
<td>21</td>
<td>24</td>
<td>9</td>
<td>9.25</td>
<td>Hull Swell  Full Sail</td>
<td>Delight</td>
<td>Fresh breeze</td>
<td>vrij krachtige</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>22</td>
<td>27</td>
<td>31</td>
<td>11</td>
<td>11.38</td>
<td>Delight tinged with anxiety</td>
<td>Strong breeze</td>
<td>krachtige</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>28</td>
<td>33</td>
<td>38</td>
<td>14</td>
<td>14.44</td>
<td>Anxiety tinged with fear</td>
<td>Moderate Gale</td>
<td>harde</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>34</td>
<td>40</td>
<td>46</td>
<td>18</td>
<td>17.65</td>
<td>Fear tinged with terror</td>
<td>Gale</td>
<td>stormachtige</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>41</td>
<td>47</td>
<td>54</td>
<td>21</td>
<td>19.78</td>
<td>Great terror</td>
<td>Strong Gale</td>
<td>storm</td>
<td></td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>48</td>
<td>55</td>
<td>63</td>
<td>25</td>
<td>22.16</td>
<td>Panic</td>
<td>Whole Gale</td>
<td>zware storm</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>11</td>
<td>56</td>
<td>63</td>
<td>75</td>
<td>29</td>
<td>24.63</td>
<td>I want my mummy!!</td>
<td>Storm</td>
<td>zeer zware storm</td>
<td></td>
<td>11</td>
</tr>
<tr>
<td>12</td>
<td>above 63</td>
<td>above 75</td>
<td>above 33</td>
<td>above 120</td>
<td>above 630</td>
<td>above 630</td>
<td>above 630</td>
<td>above 630</td>
<td>above 630</td>
<td>above 630</td>
</tr>
</tbody>
</table>
Figure 5.40: Sea State in Relation to Beaufort Wind Force Scale
5.5.1 Single Storm

The previous sections have shown that an entire storm can be characterized by just two numbers: one related to the wave period and one to the wave height. It now becomes important to predict these values from other data - such as geographical and meteorological information.

Wave Prediction

In general, prediction models are based upon the following parameters:

- **wind speed**,  
- distance over which the waves have travelled under that wind field, called **fetch** and  
- **duration** of the wind speed.

Coastal engineers often add fourth parameter: the water depth. This is less important for offshore engineering applications and will not be included here. Figure 5.41 shows the relevant parameter relations for deep water.

![Parameter Relations of a Fetch-Limited Fully Developed Sea](image)

**Figure 5.41: Parameter Relations of a Fetch-Limited Fully Developed Sea**

Significant wave heights (m) are to be read off along the logarithmic scale on the left. One can enter the diagram with a wind speed (m/s) along the vertical axis on the right and the fetch (km) chosen from the values written along the upper curve. One then reads off three remaining items:
• **Wave Height** $H$ in meters, on the left of the figure,
• **Duration** $t$ in hours, needed to generate the wave under the chosen conditions on the bottom of the figure and
• **Wave Period** $T$ in seconds, by interpolating between the dashed lines.

Most oceanographers consider a fully developed sea to be one in which - for a given wind speed - the remaining wave conditions (height and period) are no longer influenced by either the storm duration or fetch length and thus one's location. Even if one were to travel around the globe with the constant wind field, one would find that the wave height no longer increased. Fully developed sea conditions are represented in this figure by the triangular area on the right in which the wave height (for a given wind speed) is indeed independent of the duration or the fetch.

Suppose, as an exercise with figure 5.41, a wind speed of 10 m/sec (Beaufort force 5). With a fetch of 60 km, the sea no longer increases after 6 hours. This sea is defined by a significant wave height of 1.5 meters with an average wave period of 4.8 seconds. With a fetch of 600 km, the sea no longer increases after 40 hours. This sea is defined by a significant wave height of 2.0 meters with an average wave period of 6.4 seconds. Notice that, if one were to wait longer at a given location than the time duration found in this figure, the wave height would not increase further even though the waves are not oceanographically fully developed; they are limited in this case by the fetch.

**Storm Wave Data**

The table below, for "Open Ocean Areas" and "North Sea Areas" gives an indication of an average relationship between the Beaufort wind scale (or the associated average wind velocity) at 19.5 meters above the sea and the significant wave height $H_{1/3}$ and the average wave periods $T_1$ and $T_2$, defined before. These data have been plotted in figure 5.42.

<table>
<thead>
<tr>
<th>Scale of Beaufort</th>
<th>Wind Speed at 19.5 m above sea</th>
<th>Open Ocean Areas (Bretschneider)</th>
<th>North Sea Areas (JONSWAP)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(kn)</td>
<td>$H_{1/3}$ (m)</td>
<td>$T_1$ (s)</td>
</tr>
<tr>
<td>1</td>
<td>2.0</td>
<td>1.10</td>
<td>5.80</td>
</tr>
<tr>
<td>2</td>
<td>5.0</td>
<td>1.20</td>
<td>5.90</td>
</tr>
<tr>
<td>3</td>
<td>8.5</td>
<td>1.40</td>
<td>6.00</td>
</tr>
<tr>
<td>4</td>
<td>13.5</td>
<td>1.70</td>
<td>6.30</td>
</tr>
<tr>
<td>5</td>
<td>19.0</td>
<td>2.15</td>
<td>6.50</td>
</tr>
<tr>
<td>6</td>
<td>24.5</td>
<td>2.90</td>
<td>7.20</td>
</tr>
<tr>
<td>7</td>
<td>30.5</td>
<td>3.75</td>
<td>7.80</td>
</tr>
<tr>
<td>8</td>
<td>37.0</td>
<td>4.90</td>
<td>8.40</td>
</tr>
<tr>
<td>9</td>
<td>44.0</td>
<td>6.10</td>
<td>9.00</td>
</tr>
<tr>
<td>10</td>
<td>51.5</td>
<td>7.45</td>
<td>9.60</td>
</tr>
<tr>
<td>11</td>
<td>58.5</td>
<td>8.70</td>
<td>10.10</td>
</tr>
<tr>
<td>12</td>
<td>$&gt;64.0$</td>
<td>10.25</td>
<td>10.30</td>
</tr>
</tbody>
</table>
This table and figure 5.42 show lower and shorter waves in North Sea Areas, when compared at the same wind strength (Beaufort number) with those in Open Ocean Areas. Other open ocean definitions for the North Atlantic and the North Pacific, obtained from [Bales, 1983] and adopted by the 17th ITTC (1984) as a reliable relationship for these areas, are given in the table below. The modal or central periods in this table correspond with the peak period, $T_p$, as defined before.
### Open Ocean Annual Sea State Occurrences from Bales (1983) for the North Atlantic and the North Pacific

<table>
<thead>
<tr>
<th>Sea State Number (−)</th>
<th>Significant Wave Height $H_{1/3}$ (m)</th>
<th>Sustained Wind Speed 1) (kn)</th>
<th>Probability of Sea State (%)</th>
<th>Modal Wave Period $T_p$ (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Range</td>
<td>Mean</td>
<td>Range</td>
<td>Mean</td>
</tr>
<tr>
<td>North Atlantic</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 - 1</td>
<td>0.0 - 0.1</td>
<td>0.05</td>
<td>0 - 6</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>0.1 - 0.5</td>
<td>0.3</td>
<td>7 - 10</td>
<td>8.5</td>
</tr>
<tr>
<td>3</td>
<td>0.50 - 1.25</td>
<td>0.88</td>
<td>11 - 16</td>
<td>13.5</td>
</tr>
<tr>
<td>4</td>
<td>1.25 - 2.50</td>
<td>1.88</td>
<td>17 - 21</td>
<td>19</td>
</tr>
<tr>
<td>5</td>
<td>2.5 - 4.0</td>
<td>3.25</td>
<td>22 - 27</td>
<td>24.5</td>
</tr>
<tr>
<td>6</td>
<td>4 - 6</td>
<td>5.0</td>
<td>28 - 47</td>
<td>37.5</td>
</tr>
<tr>
<td>7</td>
<td>6 - 9</td>
<td>7.5</td>
<td>48 - 55</td>
<td>51.5</td>
</tr>
<tr>
<td>8</td>
<td>9 - 14</td>
<td>11.5</td>
<td>56 - 63</td>
<td>59.5</td>
</tr>
<tr>
<td>&gt;8</td>
<td>&gt;14</td>
<td>&gt;14</td>
<td>&gt;63</td>
<td>&gt;63</td>
</tr>
<tr>
<td>North Pacific</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>0 - 1</td>
<td>0.0 - 14</td>
<td>0.05</td>
<td>0 - 6</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>0.1 - 0.5</td>
<td>0.3</td>
<td>7 - 10</td>
<td>8.5</td>
</tr>
<tr>
<td>3</td>
<td>0.50 - 1.25</td>
<td>0.88</td>
<td>11 - 16</td>
<td>13.5</td>
</tr>
<tr>
<td>4</td>
<td>1.25 - 2.50</td>
<td>1.88</td>
<td>17 - 21</td>
<td>19</td>
</tr>
<tr>
<td>5</td>
<td>2.5 - 4.0</td>
<td>3.25</td>
<td>22 - 27</td>
<td>24.5</td>
</tr>
<tr>
<td>6</td>
<td>4 - 6</td>
<td>5.0</td>
<td>28 - 47</td>
<td>37.5</td>
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<td>7</td>
<td>6 - 9</td>
<td>7.5</td>
<td>48 - 55</td>
<td>51.5</td>
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<td>8</td>
<td>9 - 14</td>
<td>11.5</td>
<td>56 - 63</td>
<td>59.5</td>
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<tr>
<td>&gt;8</td>
<td>&gt;14</td>
<td>&gt;14</td>
<td>&gt;63</td>
<td>&gt;63</td>
</tr>
</tbody>
</table>

**Note:**

1) Ambient wind sustained at 19.5 m above surface to generate fully-developed seas. To convert to another altitude $h_2$, apply $V_2 = V_1 \cdot \left(h_2/19.5\right)^{1/7}$.

2) Minimum is 5 percentile and maximum is 95 percentile for periods given wave height range.

3) Based on periods associated with central frequencies included in Hindcast Climatology.

The above tables for the North Atlantic and North Pacific Oceans also include information on the long term climatology of the waves. This will be discussed in the next section.
Notice that all the methods in this section actually link wave data to wind data. This is quite common since wind data is often much more available than wave data or can be predicted rather easily from other available meteorological data.

5.5.2 Long Term

Longer term wave climatology is used to predict the statistical chance that a given wave-sensitive offshore operation - such as lifting a major top-side element into place - will be delayed by sea conditions which are too rough. Chapter 11 will be devoted entirely to various applications of this general topic in a broad sense. The current section treats the necessary input data on wave climate.

In general, wave climatology often centers on answering one question: What is the chance that some chosen threshold wave condition will be exceeded during some interval - usually days, weeks or even a year? To determine this, one must collect - or obtain in some other way such as outlined in the previous section - and analyze the pairs of data \((H_{1/3} \text{ and } T)\) and possibly even including the wave direction, \(\mu\), as well) representing each 'storm' period.

Wave Scatter Diagram

Sets of characteristic wave data values can be grouped and arranged in a table such as that given below based upon data from the northern North Sea. A 'storm' here is an arbitrary time period - often of 3 or 6 hours - for which a single pair of values has been collected.

The number in each cell of this table indicates the chance (on the basis of 1000 observations in this case) that a significant wave height (in meters) is between the values in the left column and in the range of wave periods listed at the top and bottom of the table. Figure 5.43 shows a graph of this table.

<table>
<thead>
<tr>
<th>(H_{1/3} \text{ (m)})</th>
<th>4.5</th>
<th>5.0</th>
<th>5.5</th>
<th>6.0</th>
<th>6.5</th>
<th>7.0</th>
<th>7.5</th>
<th>8.0</th>
<th>8.5</th>
<th>9.0</th>
<th>9.5</th>
<th>10.0</th>
<th>10.5</th>
<th>11.0</th>
<th>11.5</th>
<th>12.0</th>
<th>12.5</th>
<th>13.0</th>
<th>13.5</th>
<th>(\sum_{row})</th>
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<tbody>
<tr>
<td>&gt;12</td>
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<td>11.5-12.0</td>
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<td>11.0-11.5</td>
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<td>10.5-11.0</td>
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<td>10.0-10.5</td>
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<td>9.5-10.0</td>
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Wave Climate Scatter Diagram for Northern North Sea

Note: 0+ in this table indicates that less than 0.5 observation in 1000 was recorded for the given cell.
This scatter diagram includes a good distinction between sea and swell. As has already been explained early in this chapter, swell tends to be low and to have a relatively long period. The cluster of values for wave heights below 2 meters with periods greater than 10 seconds is typically swell in this case.

A second example of a wave scatter diagram is the table below for all wave directions in the winter season in areas 8, 9, 15 and 16 of the North Atlantic Ocean, as obtained from Global Wave Statistics.

<table>
<thead>
<tr>
<th>$H_s$ (m)</th>
<th>3.5 4.5 5.5 6.5 7.5 8.5 9.5 10.5 11.5 12.5 13.5</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0 0 0 0 2 30 154 362 466 378 202</td>
<td>1586</td>
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<tr>
<td>3.5</td>
<td>0 0 0 0 3 33 145 293 322 219 104</td>
<td>1116</td>
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<tr>
<td>5.5</td>
<td>0 0 0 0 7 72 289 539 548 345 149</td>
<td>1949</td>
</tr>
<tr>
<td>7.5</td>
<td>0 0 0 0 1 41 363 1200 1852 1579 843</td>
<td>6189</td>
</tr>
<tr>
<td>9.5</td>
<td>0 0 0 0 4 109 845 2485 3443 2648 1283</td>
<td>11249</td>
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<tr>
<td>11.5</td>
<td>0 0 0 0 32 295 1996 5157 6323 4319 1882</td>
<td>28576</td>
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<tr>
<td>13.5</td>
<td>0 0 0 0 41 818 4723 10537 11242 6755 2594</td>
<td>37413</td>
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<tr>
<td>15.5</td>
<td>0 0 0 0 1 138 2373 10967 20620 18718 9465</td>
<td>66371</td>
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<tr>
<td>17.5</td>
<td>0 0 0 0 7 471 6187 24075 36948 27702 11969</td>
<td>111432</td>
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<tr>
<td>19.5</td>
<td>0 0 1 31 1596 15757 47872 56347 35359 11710</td>
<td>160244</td>
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<tr>
<td>21.5</td>
<td>0 4 681 13441 56474 77259 45813 15962 2725 381 41</td>
<td>218354</td>
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<tr>
<td>23.5</td>
<td>0 10 2699 33324 47839 34532 15554 2208 262 27 2</td>
<td>122467</td>
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<td>25.5</td>
<td>5 350 5314 8131 5858 1588 216 18 1 0 0 0 41</td>
<td>37413</td>
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<td>Total</td>
<td>5 394 6881 92136 140744 217722 258541 190161 61248 19271 1864 999996</td>
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</table>

These wave scatter diagrams can be used to determine the long term probability for storms exceeding certain sea states. Each cell in this table presents the probability of occurrence of its significant wave height and zero-crossing wave period range. This probability is equal to the number in this cell divided by the sum of the numbers of all cells in the table, for instance:

$$\Pr \{ 4 < H_{1/3} < 5 \text{ and } 8 < T_2 < 9 \} = \frac{47072}{999996} = 0.047 = 4.7\%$$

For instance, the probability on a storm with a significant wave height between 4 and 6 meters with a zero-crossing period between 8 and 10 seconds is:

$$\Pr \{ 3 < H_{1/3} < 5 \text{ and } 8 < T_2 < 10 \} = \frac{47072 + 56347 + 74007 + 64809}{999996} = 0.242 = 24.2\%$$

The probability for storms exceeding a certain significant wave height is found by adding the numbers of all cells with a significant wave height larger than this certain significant wave height and dividing this number by the sum of the numbers in all cells, for instance:

$$\Pr \{ H_{1/3} > 10 \} = \frac{6189 + 3449 + 1949 + 1116 + 1586}{999996} = 0.014 = 1.4\%$$

Note that the above scatter diagram is based exclusively on winter data. Such diagrams are often available on a monthly, seasonal or year basis. The data in these can be quite different; think of an area in which there is a very pronounced hurricane season, for example. Statistically, the North Sea is roughest in the winter and smoothest in summer.
5.5.3 Statistics

Just as with waves in an individual storm, one often wants to estimate the storm intensity (wave characteristics in this case) that one should associate with some chosen and very low chance of exceedance. To do this one will usually have to extrapolate the wave data collected as in the scatter diagrams above. Two statistical distribution can be used to do this; they are described individually here.

Semi-Logarithmic Distribution

It has been found empirically that storm data - such as that presented in the wave scatter diagram for the North Sea, above, behave quite nicely if the wave height is plotted versus the logarithm of the chance of exceedance. The processing of the data in that scatter diagram is very analogous to that for waves in a single storm given much earlier in this chapter.

The plot resulting from the North Sea is given in figure 5.43. The graph is pretty much a straight line for lower probabilities, that would be needed to predict extreme events by linear extrapolation. Obviously this segment of the graph can be extrapolated to whatever lower probability one wishes to choose. This can be done using the formula:

$$\log \{ (P(H)) \} = \frac{1}{a} H$$

in which $a$ is related to the slope of the curve; see figure 5.43.

![Figure 5.43: Logarithmic Distribution of Data from Northern North Sea](image)

Weibull Distribution

A generalization of the exponential distribution given above - the Weibull distribution - is often used, too. In equation form:
Figure 5.44: Histogram and Log-Normal and Weibull Distributions

\[ P(H) = \exp \left\{ - \left( \frac{H - c}{a} \right)^b \right\} \]  

(5.130)

in which \( a \) is a scaling parameter (m), \( b \) is a fitting parameter (-) and \( c \) is a lower bound for \( H \) (m).

Figure 5.44-a shows a histogram of wave heights. A log-normal and a Weibull fit to these data are shown in figures 5.44-b and 5.44-c.

In many cases the value of \( b \) in the Weibull equation above is close to 1. If - as is usually the case with waves - \( c = 0 \) and \( b \) is exactly 1, then the Weibull distribution reduces to the log-normal distribution given earlier.